

Applications of the Lattice Method to Infinite Dimensional Hermitean Spaces

Habilitationsschrift

zur Erlangung der Venia legendi
vorgelegt der
Philosophischen Fakultät II
der Universität Zürich

von
WERNER BÄNI
aus Ürkheim AG

Zürich 1981
Zentralstelle der Studentenschaft



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INTRODUCTION

Mapping problems in vector spaces are found in nearly every domain of mathematics and are therefore familiar to every mathematician. In most cases the formulation of such problems does not require any restrictions on the dimension of the spaces involved. However, the methods for constructing the desired maps as well as the results obtained show a large discrepancy between finite and infinite dimensional spaces. When passing from finite to infinite dimensions the inductive arguments used for finite dimensional spaces are replaced by recursive constructions, which are of course based on the finite dimensional situation. Therefore it is not surprising that spaces of countably infinite dimension are much better accessible than spaces of uncountable dimension¹⁾: An isomorphism $T: E \rightarrow E'$ between the spaces E, E' of countable dimension may be obtained as the union of a nested sequence $T_1: X_1 \rightarrow X'_1$ of isomorphisms between finite dimensional subspaces X_1, X'_1 of E, E' . This very case we wish to discuss in the present notes.

Our purpose is to demonstrate the methods which have been developed over the last few years ([11], [12], [3]) and establish some possible variations and extensions. This will be done by solving a specific nontrivial problem, namely, Witt's problem in hermitean spaces (E, Φ) of countable dimension over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (i.e., the problem of classifying subspaces of such spaces with respect to isometries of the whole spaces). Related questions are also considered.

Before giving a brief description of our results and their connection with papers already published (and thereby describing our motivation for tackling this particular problem) we would like to have a closer look at the methods used. It is our conviction that they are of interest in a broader mathematical context (affine mapping problems, projective mapping problems etc.).

The isomorphism $T: E \rightarrow E'$ which is to be constructed must have additional properties according to the specific problem. A frequently occurring condition is that it should map certain subspaces of E onto corresponding subspaces of E' . It is then obvious to consider the

¹⁾ Here (and throughout the whole book) we are dealing with purely algebraic situations. 'Dimension' means vector space dimension in the sense of linear algebra. Topologies may also occur, but merely as an auxiliary device.

sublattice of $L(E)$ (the lattice of all linear subspaces of E) generated by these distinguished subspaces. However, in most situations it is not at all evident which lattice one should consider and what it looks like; indeed, the most difficult part of the task often consists in defining and computing a suitable lattice; it should contain enough information without getting too unwieldy. Once the lattice is at hand it may be used in two ways: it allows the definition of invariants that are relevant to the situation (simplest example: codimensions between neighbouring lattice elements) and furthermore, it serves as a scaffold for the construction of the desired isomorphism T . By this we mean that the members $T_1: X_1 \rightarrow X'_1$ of the nested sequence mentioned earlier are constructed in such a way that $T_1(X_1 \cap A) = X'_1 \cap A'$ is fulfilled for each pair A, A' of corresponding lattice elements in E, E' (and further technical conditions apply). More generally, the problem consists in specifying a nonempty collection G of isomorphisms $t: X \rightarrow X'$ between finite dimensional subspaces X, X' of E, E' such that $t(X \cap A) = X' \cap A'$ (where A, A' are as above) and the following "Ping-Pong property" is satisfied

(PP) If $(t: X \rightarrow X') \in G$ and $y \in E \setminus X$ (or $y' \in E' \setminus X'$) then there is $(t_1: X_1 \rightarrow X'_1) \in G$ such that $y \in X_1 \supset X$ (or $y' \in X'_1 \supset X'$) and $t_1|_X = t$.

We have detailed the general features of this lattice method (as used here) in I.3 below.

Let us now pass on to our specific problem. Witt's Theorem [22] states that each isometry $T: V \rightarrow V'$ between two subspaces of a finite dimensional nondegenerate quadratic space (E, ϕ) (in characteristic $\neq 2$) extends to a metric automorphism of E . The failure of this statement for infinite dimensions is obvious and raises the following two fundamental problems:

1) If we are given subspaces V, V' of E , in which cases can we define an isometric automorphism T of E such that $TV = V'$? In other words, we are to describe the set of congruence classes of subspaces $V \subseteq E$, i.e. the set of orbits of the orthogonal group of (E, ϕ) under its natural action on the set of subspaces $V \subseteq E$.

2) If we are given an isometry $T_0: V \rightarrow V'$, under what conditions does it extend to an element of the orthogonal group of (E, ϕ) ? (T_0 may not extend although V and V' are congruent.)

We shall refer to these problems as the "congruence problem" and the "extension problem". In infinite dimensions they prove to be quite disjoint topics. Investigations in these directions have been carried out by I. Kaplansky [14] and mainly by H. Gross [9], [10], [11], [12]. In the present notes we are concerned with hermitean spaces (E, ϕ) over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H} = \begin{pmatrix} -1 & -1 \\ \mathbb{R} \end{pmatrix}$ of countable dimension (hence there is no overlapping with the theory of Hilbert -, Pontrjagin -, and Krein spaces [6]). In order to explain our restriction concerning the base field it is necessary to recall some of the facts already known. In doing so we assume E to be of dimension $\aleph_0^{1)}$ over a commutative base field and ϕ to be symmetric nondegenerate.

In the case of a symmetric space (E, ϕ) over a quadratically closed field k the congruence problem has a definite and satisfactory answer ([9]): A complete system of invariants of the orbit of a subspace V is formed by the codimensions between neighbouring spaces in Kaplansky's lattice ([14]), the smallest lattice of subspaces of E containing $0, V, E$ and, along with each X , its orthogonal complement $X^\perp$ (such a lattice is called $\perp$ -stable). This system is made up of seven cardinal numbers $\leq \aleph_0$. If one drops the restriction on k the problem gets more involved and strongly dependent on the ground field. Gross [12] (Chapt. XII) and Schneider [21] have treated the situation in which (E, ϕ) is a positive definite space over a suitable ordered field $(k, <)$. They obtained statements applying to all subspaces V with the restriction that the ordering of k is archimedean (i.e., $(k, <) \subset (\mathbb{R}, <)$). In this case the orbit of V has been characterized by (at most) four $\mathbb{R}$ -matrices of (possibly) infinite size, subject to a complicated equivalence relation. For general subspaces V the congruence problem was not simplified by this reformulation, but in the case of $\perp$ -dense subspaces ($V^\perp = 0$) it permitted sufficient insight: For $\perp$ -dense subspaces V the congruence problem turned out to be equivalent to the problem of classifying symmetric k -bilinear positive maps $E/V \times E/V \rightarrow \mathbb{R}$ with respect to "isometry", a trivial problem if $k = \mathbb{R}$, and a hopeless problem (cf. Appendix) if $k \not\subset \mathbb{R}$. Therefore a reasonable classification of all subspaces of a space (E, ϕ) over an orderable base field can at best be expected in the case of $k = \mathbb{R}$.

1) Recently, H. Gross has been able to extend his results in [11] to spaces (E, ϕ) of uncountable dimension, provided that (E, ϕ) has an orthogonal decomposition into spaces of finite dimension. We refer to a forthcoming paper.

These observations have led us to the present investigations. We have already published the special case of a positive definite $\mathbb{R}$ -space (E, ϕ) in [2]. However, many interesting problems do not arise except in indefinite spaces (as, e.g., the problem of classifying the maximal positive definite subspaces). Passing from definite to general $\mathbb{R}$ -spaces raises a considerable amount of new difficulties, as Gross already pointed out in [10] and [12] (Chapt. XIII) where he dealt with dense subspaces of finite codimension. We may further illustrate the new difficulties by mentioning that the computation of the relevant lattice occupies the main body of Chapt. III below whereas in [2] ((E, ϕ) positive definite) it was trivial. Hence the benefit of the lattice method manifests itself much better in the more general situation presented here. We further point out that our example is the first in this class of problems which required a detailed analysis of an infinite lattice.

As to the congruence problem, it follows that the orbits of a large class of subspaces (including all dense subspaces, all closed subspaces ($V^{\perp\perp} = V$), all maximal positive definite subspaces (provided that (E, ϕ) is of finite Witt index); cf. Chapt. V for further examples) may be characterized by suitable cardinal number invariants. Nevertheless, a complete classification of all subspaces remains, in a rather precise sense, inconceivable (cf. Appendix).

We have also used the lattice method to obtain results concerning the extension problem. Related results in other situations - and proved by a different method - can be found in [1], [12] (Chapt. X), [21] (Satz VI.8), and [17] (4.2). As an application of our extension theorem we have given a criterion for a pair (A, B) of subspaces to be orthogonally separated in E , i.e., $E = E_1 \oplus E_2$ such that $A \subset E_1$, $B \subset E_2$.

In the last chapter we prove some congruence theorems over orderable base fields other than $\mathbb{R}$. Thereby we hope to make clear that the restriction to $\mathbb{R}$ (or $\mathbb{C}, \mathbb{H}$, for hermitean spaces) as ground field corresponds to the actual situation and has not been forced upon us by inappropriate concepts and methods.

The author is indebted to H. Gross for a great number of useful and stimulating discussions on the subject and for many valuable suggestions concerning the manuscript.

CHAPTER I

PRELIMINARIES

1. Notations

Concepts and notations that are not explained may be found in Gross' book [12]. We let (E, Φ) be a nondegenerate hermitean K -space of dimension at most $\aleph_0$, where K , with the exception of a few cases, is one of the fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, endowed with its usual involution $\lambda \mapsto \lambda^*$. Thus, every subspace $V \subset E$ admits a decomposition (a so-called fundamental decomposition)

$$V = R \oplus V_- \oplus V_+$$

with $R = V \cap V^\perp$ the radical of V , V_- negative definite and V_+ positive definite. The cardinal numbers

$$n^+(V) = n^+(V, \Phi) := \dim V_+, \quad n^-(V) = n^-(V, \Phi) := \dim V_-$$

are independent of the decomposition and, together with $\dim R$ determine the isometry type of V . If $n^+(V) = n^-(V)$ and $\dim R = 0$ the subspace V is termed hyperbolic (being a sum of hyperbolic planes).

Such a triple $(E, \Phi, V) = E$ with $V \subset E$ is called an embedding. Any adjective belonging to the noun "embedding" refers to the subspace V ; thus we speak about nondegenerate embeddings ($V \cap V^\perp = 0$), $\perp$ -dense embeddings ($V^\perp = 0$), $\perp$ -closed embeddings ($V = V^{\perp\perp}$) and so on. Two embeddings E, E' are called congruent, $E \cong E'$, if there is an isometry $T: (E, \Phi) \longrightarrow (E', \Phi')$ with $TV = V'$. Orthogonal sums of countable families of embeddings are formed in the obvious way.

2. The ghost form

2.1 For every embedding (E, ϕ, V) we define a subspace $C(V)$ of E (denoted more exactly by $C(E, \phi, V)$ if there is any risk of confusion) and a hermitean form ϕ_V on $C(V)$, called the ghost form of V , as follows.

a) If $n^-(V) = n^+(V) = \infty$ let $C(V) = V + V^\perp$ and

$$\phi_V(v+w, v'+w') = \phi(w, w') \quad \text{for } v, v' \in V, w, w' \in V^\perp.$$

b) If $\min(n^-(V), n^+(V)) < \infty$ choose any orthogonal basis ¹⁾ $(v_i)_{i \in I}$ of any supplement of R in V and set

$$C(V) := \{x \in R^\perp \mid \sum_{i \in I} \phi(x, v_i) \phi(v_i, v_i)^{-1} \phi(v_i, x) \text{ exists in } R\}$$

and for $x, y \in C(V)$:

$$\phi_V(x, y) := \phi(x, y) - \sum_{i \in I} \phi(x, v_i) \phi(v_i, v_i)^{-1} \phi(v_i, y).$$

It follows from the Schwarz inequality that $C(V)$ is a subspace and that the sum in the definition of ϕ_V exists. That $C(V)$ and ϕ_V do not depend on the choice of the v_i can be seen as follows. Firstly, if $n^+(V) < \infty$ and $n^-(V) = \infty$ we have $C(V) = R^\perp = V + V^\perp$ and for ϕ_V we get the same formula as in a) of the definition. Hence we may assume w.l.o.g. that $n^+(V) = \infty$, $n^-(V) < \infty$. Secondly, if $n^-(V) = 0$, i.e. if the supplement V_0 of R in V spanned by the v_i is positive definite and thus carries the norm $v \mapsto \|v\| = \phi(v, v)^{\frac{1}{2}}$, then the sum

$\sum_{i \in I} \phi(x, v_i) \phi(v_i, v_i)^{-1} \phi(v_i, x)$ can easily be seen to coincide with the square of the norm of the linear functional $V_0 \rightarrow K, v \mapsto \phi(v, x)$.

From what we have said so far we infer that $C(V)$ and ϕ_V are not

1) a basis in the algebraic sense, i.e. a so-called Hamelbasis. The word "basis" will always be used in this sense.

affected by a change of bases in V_0 if this change actually takes place either in a finite dimensional or in a positive definite orthogonal summand of V_0 . Now, any change of (orthogonal) bases in V_0 may be decomposed into a number of changes of this particular kind (since $n^-(V) < \infty$). Finally, because $C(V) \subset R^\perp$, the choice of the supplement V_0 of R in V is irrelevant.

The following often used fact is immediate from the definition of Φ_V :

$$(1) \quad x \in C(V), y \in V^\perp \Rightarrow \Phi_V(x, y) = \Phi(x, y).$$

Denoting by $r\Phi_V$ the radical of Φ_V , $r\Phi_V = \{x \in C(V) \mid \Phi_V(x, y) = 0 \ \forall y \in C(V)\}$, we get furthermore

$$(2) \quad V \subset r\Phi_V \subset V^{\perp\perp}.$$

Occasionally we shall use the suggestive notation $\Phi V \Phi$ for the form $\Phi|_{C(V) \times C(V)} - \Phi_V$ on $C(V)$, which depends only on $\Phi|_{E \times V}$ and $\Phi|_{V \times V}$,

$$(\Phi V \Phi)(x, y) = \sum_1 \Phi(x, v_1) \Phi(v_1, v_1)^{-1} \Phi(v_1, y) \quad \text{for } x, y \in C(V).$$

(The formula holds in both cases, a) and b), of the definition.)

It should be noted that what we have done so far remains perfectly valid if the assumption of (E, Φ) being nondegenerate is abandoned.

The isometry types of (E, Φ) , (V, Φ) and $(C(V), \Phi_V)$ are related by

$$(3) \quad n^-(E, \Phi) = n^-(V, \Phi) + n^-(C(V), \Phi_V) + \dim E/C(V)$$

and a similar equation for n^+ .

Proof. After having discarded trivial cases, we may assume that $n^+(V, \Phi) = \infty$, $n^-(V, \Phi) < \infty$ and $\dim R < \infty$. If we pass from E to $R^\perp/R$ both sides of (3) decrease by the summand $\dim R$, and if from E we

orthogonally split off a (maximal negative definite) subspace V_- of V we have to subtract $n^-(V, \Phi)$ from both sides of (3). Thus we are left with the case where V is positive definite. If now Φ is negative definite on $B \subseteq E$ then Φ_V is so on $B \cap C(V)$ and $\dim B/B \cap C(V) \leq \dim E/C(V)$, whence " $\leq$ " is valid in (3). For the reverse inequality, consider finite dimensional subspaces $A_1 \subset C(V)$ on which Φ_V is negative definite (hence $A_1 \cap V = 0$) and finite dimensional subspaces $A_2 \subset E$ with $A_2 \cap C(V) = 0$. For every such $A = A_1 + A_2$ we shall construct $B \subseteq E$ on which Φ is negative definite, and such that $V \oplus B = V \oplus A$. Obviously this will prove " $\geq$ " in (3). Choose any orthonormal basis $(v_i)_N$ of V and consider the monotonically decreasing sequence ρ_n of continuous functions $A \rightarrow \mathbb{R}$, defined by

$$\rho_n(x) := \Phi(x, x) - \sum_{i=1}^n \Phi(x, v_i) \Phi(v_i, x) = \Phi(x - \sum_{i=1}^n \Phi(x, v_i) v_i, x - \sum_{i=1}^n \Phi(x, v_i) v_i)$$

(let the topology on A be given by an arbitrary positive definite form Ψ on A). For $0 \neq x \in A_1$ we have $\lim_{n \rightarrow \infty} \rho_n(x) = \Phi_V(x, x) < 0$ whereas for $x \in A \setminus A_1$ the sequence $\rho_n(x)$ tends to $-\infty$. Since the Ψ -unitsphere in A is compact, there is $n \in \mathbb{N}$ such that $\rho_n(x) < 0$ for $0 \neq x \in A$. Hence we may choose $B = \{x - \sum_{i=1}^n \Phi(x, v_i) v_i \mid x \in A\}$.

As a consequence of (3), we see that if (E, Φ) is positive definite then $C(V) = E$ and $(E, \Phi_V)/r\Phi_V$ is again positive definite. Of course this would already be clear from the equality in the definition of ρ_n above.

2.2 There is another way of interpreting the ghost form which seems worth mentioning (we shall not use it, so we skip the proofs). Let us assume not only that $n^-(V) < \infty$ but also $n^-(E) < \infty$, which is no essential restriction since $C(V)$ and Φ_V are defined "locally over V ". We may put a norm on E by choosing a fundamental decomposition

$E = E_- \overset{\perp}{\oplus} E_+$ and endowing E_- and E_+ with the norms induced by $-\phi$ on E_- and ϕ on E_+ . The norm topology τ thus defined on E does not depend on the choice of the decomposition. Our form ϕ extends continuously to a form $\tilde{\phi}$ on the completion $(\tilde{E}, \tilde{\tau})$ of (E, τ) , and $(\tilde{E}, \tilde{\phi})$ is a Pontrjagin space in the sense of [6], i.e. an orthogonal sum of a Hilbert space and a finite dimensional space. In $\tilde{E}$ every nondegenerate $\tilde{\tau}$ -closed subspace is even an orthogonal summand. Thus, if $\tilde{V}$ is the $\tilde{\tau}$ -closure of V in $\tilde{E}$ and the symbol $\perp$ denotes orthocomplementation in $(\tilde{E}, \tilde{\phi})$ we get a decomposition

$$\tilde{E} = (R \oplus R') \overset{\perp}{\oplus} A \overset{\perp}{\oplus} B$$

with $R = \tilde{V} \cap \tilde{V}^\perp$, $\dim R' = \dim R < \infty$, $\tilde{V} = R + A$, $R^\perp = R + A + B$. Now it can be shown that

$$(4) \quad \begin{aligned} C(V) &= E \cap R^\perp \\ \phi_V(x, y) &= \tilde{\phi}(x_B, y_B) \quad \text{for } x, y \in C(V) \end{aligned}$$

with x_B, y_B the projections of x, y in B . This gives us the desired interpretation.

Using (4) it is not too difficult to derive the following interesting facts. Let $V \subset W \subset E$ and, say, $n^-(W) < \infty$, but drop the restriction on $n^-(E)$. We have

$$(5) \quad \begin{aligned} C(W) &= W + C(W \cap C(V)) \\ \phi_W(w+x, w'+x') &= \phi_{W \cap C(V)}(x, x') \quad \text{for } w, w' \in W; x, x' \in C(W \cap C(V)) \end{aligned}$$

and furthermore

$$(6) \quad \begin{aligned} C(W \cap C(V)) &= C(C(V), \phi_V, W \cap C(V)) \\ \phi_{W \cap C(V)} &= (\phi_V)_{W \cap C(V)}. \end{aligned}$$

3. On the lattice method

3.1 In the algebraic theory of infinite dimensional inner product spaces (E, Φ) many problems can be cast into the following form. There is given a lattice isomorphism $\tau: V \longrightarrow V'$ between lattices of linear subspaces of (E, Φ) , (E', Φ') , and one has to look for an isometry $T: E \longrightarrow E'$ which induces τ . The strategy is to construct such T recursively: Find a nested sequence

$$T_i: X_i \longrightarrow X'_i \quad (i \in \mathbb{N})$$

of isometries between finite dimensional subspaces X_i, X'_i , compatible with τ (i.e. $T_i(X_i \cap A) = X'_i \cap \tau A$ for all $A \in V$) and such that T_{i+1} extends T_i , $E = \bigcup X_i$, $E' = \bigcup X'_i$ (we assume that $\dim E = \dim E' = \aleph_0$). The isometry T defined by $T|_{X_i} = T_i$ will then certainly map each $A \in V$ onto τA . This recursive construction can easily be carried out provided we are given a nonempty collection G of isometries $T: X \longrightarrow X'$ ($X \subseteq E, X' \subseteq E', \dim X < \infty$) compatible with τ , with the following "Ping-Pong" property.

If $(T: X \longrightarrow X') \in G$ and $y \in E$ ($y' \in E'$) there exists
(PP) $(T_1: X_1 \longrightarrow X'_1) \in G$ such that $X \subset X_1$, $X' \subset X'_1$, $T_1|_X = T$
and $y \in X_1$ ($y' \in X'_1$).

Such a G is called a set of partial isomorphisms with respect to τ . Gross [11] succeeded in proving the existence of a set of partial isomorphisms in general situations provided the elements of the lattices contain "sufficiently many" isotropic vectors (the case, e.g., in symmetric spaces over quadratically closed fields). If the condition on isotropic vectors fails to hold, matters quickly get out of control, even in the case of such a nice field as $\mathbb{R}$. However, we shall procure a lattice suitable to attack Witt's problem in hermitean spaces over

R, C, H , and we shall exhibit a set of partial isomorphisms under appropriate conditions.

3.2 In order to avoid duplication of the proofs in [11] or [12] Chap. IV we have tried to condense the main arguments developed there into some lemmas. In what follows, (E, ϕ) and (E', ϕ') are trace-valued nondegenerate sesquilinear spaces of dimension N_0 over an arbitrary (possibly non-commutative) field k , and $V \subset L(E)$, $V' \subset L(E')$ are sublattices of the full lattices of linear subspaces of E, E' and containing E, E' respectively. The lattices are supposed to be $\perp$ -stable: $A \in V \Rightarrow A^\perp \in V$. Finally let $\tau: V \rightarrow V'$ be a lattice isomorphism which respects indices ($A, B \in V, A \subset B \Rightarrow \dim B/A = \dim \tau B/\tau A$) and commutes with $\perp$ ($A \in V \Rightarrow (\tau A)^\perp = \tau(A^\perp)$). For any $X \subset E$ of finite dimension and $y \in E$ we set

$$M = M(X, y) := \{A \in V \mid y \in X + A\}.$$

If X satisfies the condition

$$(7) \quad (X + A) \cap (X + B) = X + (A \cap B) \quad \text{for all } A, B \in V$$

then M is a filter in V . Recall that a filter $M \subset V$ is said to be prime if, for $A, B \in V$, $A + B \in M$ implies $A \in M$ or $B \in M$.

Lemma 1. Let $y \in E \setminus X$ and let X satisfy (7). If $M(X, y)$ is a prime filter then $X_1 = X \oplus ky$ satisfies (7) in lieu of X .

Note that, by [13], condition (7) is equivalent to

$$(8) \quad (X \cap A) + (X \cap B) = X \cap (A + B) \quad \text{for all } A, B \in V.$$

Put $D = D(X, y) := \bigcap M(X, y)$, which need not belong to V , and note that $y \in X + D$ by (7) (the directed set $\{X \cap A \mid A \in M\}$ has a smallest element

since $\dim X < \infty$, whence $\cap \{X + A \mid A \in M\} = X + \cap M$. Thus, to say that $D \in V$ is equivalent to saying that M is generated by a single element. E.g., if $B \in V$ and $y \in B \setminus (X + B_0)$ where $B_0 := \Sigma \{A \in V \mid A \not\subset B\}$ then $D(X, y) = B$.

In the following two lemmas $T: X \longrightarrow X'$ is an isomorphism between finite dimensional subspaces $X \subset E$, $X' \subset E'$ which is compatible with τ , i.e.

$$(9) \quad T(X \cap A) = X' \cap \tau A \quad \text{for all } A \in V.$$

Lemma 2. Let $y \in D \setminus X$, $y' \in D' \setminus X'$ and suppose that $M(X, y)$, $M(X', y')$

are generated by D, D' respectively with $D' = \tau D$. Define

$T_1: X_1 = X \oplus ky \longrightarrow X'_1 = X' \oplus ky'$ by $T_1|_X = T$ and $T_1 y = y'$. Then (9) holds for $T_1: X_1 \longrightarrow X'_1$ in lieu of $T: X \longrightarrow X'$.

Lemma 3. Let $A, B \in V$, $y \in B \setminus (X + A)$ and let $T: X \longrightarrow X'$ be an isometry which fulfills (9). Then there exists $y' \in \tau B \setminus (X' + \tau A)$ such that $\phi'(y', Tx) = \phi(y, x)$ for all $x \in X$. In the following cases y' may be chosen to satisfy, moreover, $\phi'(y', y') = \phi(y, y)$:

- a) B is totally isotropic
- b) $\tau B / \tau B \cap \tau B^\perp$ is of infinite dimension and hyperbolic.

3.3 Now let F be the collection of all isometries $T: X \longrightarrow X'$ (between finite dimensional subspaces $X \subset E$, $X' \subset E'$) which meet the requirements (7), (7') (the corresponding property of X' relative to V') and (9). In view of 3.2 it is natural, when searching for a set of partial isomorphisms, to do this inside F . Indeed, in the cases treated in [11], F itself turned out to be a set of partial isomorphisms. Suppose we have a candidate $G \subset F$ and we are to verify (PP) in 3.1.

Thus let $(T: X \rightarrow X') \in G$ and, say, $y \in E \setminus X$ be given. In the situations we shall have to consider the space $D = D(X, y)$ will always belong to $\mathcal{V}$, and we shall be able to reduce the verification of (PP) to such cases where $y \in D \setminus (X + D_0)$ with $D_0 := \Sigma \{A \in \mathcal{V} \mid A \subsetneq D\}$. In these cases we shall define $T_1: X_1 = X \oplus ky \rightarrow X'_1 = X' \oplus ky'$ by $T_1|_X = T$, $T_1 y = y'$ with a suitable vector $y' \in \tau D \setminus (X' + \tau D_0)$ (D_0 will also belong to $\mathcal{V}$), and show that $T_1 \in G$. Hereby we have no need of worrying about the conditions (7) (or (8)) and (9), thanks to Lemma 1 and Lemma 2 above. (Note that M is prime if $D \neq D_0$ and $\mathcal{V}$ is distributive.)

We conclude with another useful remark. If in the interval $[0, D]$ of $\mathcal{V}$ the minimum condition holds we shall frequently be allowed - upon an induction argument - to replace the given vector $y \in D$ by a suitable vector $y + d_0$ with $d_0 \in D_0$. Arguments of this kind will be used in I.4 and IV.3.

4. Dense embeddings

Among the embeddings investigated in the following chapters an important role is played by the dense embeddings. Therefore, and in order to illustrate the abstract considerations of the previous section we shall now prove the Congruence Theorem for dense embeddings. We presuppose, however, the fundamental extension lemma proved in Chap. II.

Theorem 1. Two $\perp$ -dense embeddings E, E' are congruent if and only if the following invariants of E agree with the corresponding invariants of E' :

- a) the isometry type of (V, Φ)
- b) the isometry type of $(C(V), \Phi_V) / r \Phi_V$
- c) the dimensions of $r \Phi_V / V$ and $E/C(V)$.

Proof . More exactly, we shall show that every isometry

$\bar{T}: (C(V), \phi_V)/r\phi_V \longrightarrow (C(V'), \phi'_V)/r\phi'_V$, can be lifted to an isometry $(E, \phi) \longrightarrow (E', \phi')$ mapping V onto V' . We choose our lattice V to be the chain $0 \subset V \subset r\phi_V \subset C(V) \subset E$. By a) b) c) the obvious lattice isomorphism $V \rightarrow V'$ respects indices. Let $G \subset F$ consist of those $(T: X \rightarrow X') \in F$ which are compatible with $\bar{T}$:

(10) The map $X \cap C(V)/X \cap r\phi_V \longrightarrow X' \cap C(V')/X' \cap r\phi'_V$, induced by T coincides with the restriction of $\bar{T}$.

To prove the Ping-Pong property, let $(T: X \rightarrow X') \in G$ and $y \in E \setminus X$ be given. Let $D = D(X, y)$ be defined as in 3.2, i.e., D is the smallest element of V with $y \in X + D$. Denoting by D_0 its predecessor in V we may assume that $y \in D \setminus (X + D_0)$, and we shall seek a match y' to y in $D' \setminus (X' + D'_0)$. If $T_1: X \oplus Ky \longrightarrow X' \oplus Ky'$ is to belong to G we must have

$$(11) \quad \phi'(y', Tx) = \phi(y, x) \quad \text{for all } x \in X,$$

$$(12) \quad \phi'(y', y') = \phi(y, y) \quad ,$$

$$(13) \quad \text{if } D \subset C(V) \text{ then } \bar{T}(y + r\phi_V) = y' + r\phi'_V .$$

Let us examine the various possibilities for D .

$D = V$: By (10) the map $T: X \cap C(V) \longrightarrow X' \cap C(V')$ is an isometry with respect to ϕ_V, ϕ'_V . Therefore, and since $C(V+X, \phi, V) = V+X \cap C(V)$, the existence of our y' follows from Lemma 1 in II.1 below.

$D \in \{r\phi_V, C(V), E\}$: We begin by hunting up a vector $y' \in D' \setminus (X' + D'_0)$ with (11) and (13). Indeed, if $D = E$ or $D = r\phi_V$ it suffices to apply Lemma 3. In case $D = C(V)$ pick y' with (13), choose $v' \in V'$ with $\phi'(y' + v', Tx) = \phi(y, x)$ for all $x \in X$ and replace y' by $y' + v'$. (The existence of such v' follows from the fundamental Lemma 8, p. 25 in [12], which we shall use freely throughout.) Now, to arrange for (12),

we would like to add to y' a vector $z' \in V' \cap X'^{\perp} \cap Y'^{\perp}$ with $\Phi'(z', z') = \Phi(y, y) - \Phi'(y', y')$. If we assume that $n^+(V) = n^+(V') = \infty$ such a correction is possible if $\Phi(y, y) \geq \Phi'(y', y')$ but may prove impossible if $\Phi(y, y) < \Phi'(y', y')$. In this case we wriggle out by observing that it suffices to solve the problem for some $y + z$ with $z \in V$. For, by the case $D = V$ already done, we may extend T in a further step from $X + K(y + z)$ to $X + K(y + z) + Kz$.

By forming orthogonal sums of embeddings of codimension 1 it is easy to construct ι -dense embeddings with arbitrarily prescribed invariants (taking into account that if V is hyperbolic then $C(V) = r\Phi_V = V$). Hence we get

Corollary. Any ι -dense embedding E admits a decomposition

$$E = \bigoplus_{\iota \in I} E_{\iota} \quad \text{such that} \quad \dim E_{\iota} / V_{\iota} = 1 \quad \text{for each } \iota \in I.$$

Some special cases of Theorem 1 may already be found in the literature (in a different terminology). Schneider [21] and Gross [12] Chap. XII assume that (E, Φ) is positive definite whereas in [10] and [12] Chap. XIII Gross considers but embeddings of finite codimension.

CHAPTER II

DENSE EMBEDDINGS OF FINITE CODIMENSION

1. The extension lemma

In view of the central importance of the extension lemma (Lemma 1 below) in our treatment of Witt's problem, a few remarks concerning its "historical" development are certainly in order. But let us state the lemma first!

Lemma 1. Let E, E' be two $\perp$ -dense embeddings with $(V, \Phi) \cong (V', \Phi')$ and let $T: X \rightarrow X'$ be an isometry between finite dimensional subspaces $X \subseteq E, X' \subseteq E'$ with $E = V + X, E' = V' + X'$, such that

$$a) \quad T(X \cap V) = X' \cap V'$$

$$b) \quad T(X \cap C(V)) = X' \cap C(V') \quad \text{and} \quad T: (X \cap C(V), \Phi_V) \longrightarrow (X' \cap C(V'), \Phi'_{V'})$$

is isometric.

Then, given $v \in V \setminus X$, there exists $v' \in V' \setminus X'$ with $\Phi'(v', Tx) = \Phi(v, x)$ for all $x \in X$ and $\Phi'(v', v') = \Phi(v, v)$.

Similar results have been established by Gross in two special cases: $n^-(E) = 0$ (see [12] Chapt. XII; what he really needs is $n^-(V) = 0$ and $C(V) = E$), and further $n^-(V) = 0$ and $\dim E/V \leq 2$ (see [12] Chapt. XIII). They proved sufficient to derive the congruence theorem for $\perp$ -dense embeddings if either $n^-(E) = 0$ or $\dim E/V < \infty$. Indeed, leaving alone the (most simple) case of a hyperbolic V , one may choose fundamental decompositions for V and V' and orthogonally split off V_- and V'_- (assuming that $n^-(V) < \infty$) at the very beginning of the proof of Theorem 1 in I.4, thus reducing the problem to the case $n^-(V) = 0$. This would prove the congruence theorem for embeddings with $\dim E/V \leq 2$. The generalization to arbitrary $\dim E/V < \infty$ (if $n^-(E) \neq 0$) was done by skilful combinatorial arguments (see loc.cit.). Such tricks

will not work when we are dealing with more general embeddings, and that is why we need Lemma 1 in its full generality.

The proof of Lemma 1 shall be divided into two stages: In the first stage (II.2) we treat the case $n^-(V) = 0$. Compared with the proofs in [12] a considerable simplification has taken place, thanks to the fact that it has been possible to avoid the use of determinants. It is due to this fact that we were able to prove Lemma 1 for arbitrary $\dim E/V < \infty$ in the case of the noncommutative base field $K = \mathbb{H}$. In the second stage (II.3) we reduce the case $n^-(V) \neq 0$ to the former case. Hereby we may assume that $n^-(V) < \infty$, since the case of a hyperbolic V is covered by I.3 Lemma 3.

2. The case $n^-(V) = 0$

2.1 Choose X_0 with $X = X \cap V \oplus X_0$, put $V = X \cap V \oplus V_0$, $v = x + v_0$ ($x \in X \cap V$, $v_0 \in V_0$) and define $E_0 := V_0 \oplus X_0$. Let $X'_0 = TX_0$ and define V'_0, E'_0 analogically. By replacing E, V, X, v etc. by E_0, V_0, X_0, v_0 etc. we see that it suffices to do the case $X \cap V = 0$, i.e. $E = V \oplus X$. Now fix any basis $x_1, \dots, x_m$ of X . For each orthonormal basis $(v_i)_{i \in \mathbb{N}}$ of V we define a sequence $(A_n)_{n \in \mathbb{N}}$ of $m \times m$ -matrices,

$$(1) \quad A_{n \vee \mu} := \sum_{i=1}^n \phi(x_v, v_i) \phi(v_i, x_\mu).$$

This is a monotonically increasing (i.e. $A_{n+1} - A_n$ is positive) sequence of positive hermitean matrices, and for large n we have that A_n is even definite. Otherwise there would exist $x \in X$ with $\phi(x, v_i) = 0$ for all i , contradicting $V^\perp = 0$. If $B_n = A_n^{-1}$ is the inverse of A_n ($n \geq n_0$) the equation

$$B_n - B_{n+1} = B_{n+1}(A_{n+1} - A_n)B_{n+1} + B_{n+1}(A_{n+1} - A_n)B_n(A_{n+1} - A_n)B_{n+1}$$

shows that the sequence $(B_n)_{n \geq n_0}$ of positive definite hermitean

matrices is monotonically decreasing and hence converges,

$$(2) \quad B := \lim_{n \rightarrow \infty} B_n.$$

We note that if $A = \lim_{n \rightarrow \infty} A_n$ exists (which is equivalent to $C(V) = E$) then $A = B^{-1}$ is the matrix of the form $\Phi - \Phi_V$ with respect to the basis $x_1, \dots, x_m$ of X . In this case it follows from I.2.1 that B is independent of the v_i chosen to define it. But this holds always.

Claim. The matrix B does not depend on the choice of the $(v_i)_{i \in \mathbb{N}}$.

Proof. Let $(\bar{v}_i)_{i \in \mathbb{N}}$ be a second orthonormal basis of V and $\bar{B}$ the resulting matrix. In order to prove that $B = \bar{B}$ it is enough to show that for each row vector $y \in K^m$ we have ¹⁾ $yBy^* \leq y\bar{B}y^*$. To this end, choose orthonormal bases $B_1, B_2, B_3, \dots$ such that

- $\alpha)$ the first i vectors of B_i are $\bar{v}_1, \dots, \bar{v}_i$
- $\beta)$ B_i differs from $(v_i)_{i \in \mathbb{N}}$ in at most finitely many vectors.

Let $A_n(i), B(i)$ be the matrices associated to B_i by (1) and (2). From $\alpha)$ it follows that $A_n(i) = \bar{A}_n$ for $n \leq i$ whereas from $\beta)$ we infer that $B(i) = B$ for all i (even $A_n(i) = A_n$ for $n \gg i$). If we had $yBy^* > y\bar{B}y^*$ there would exist $n \in \mathbb{N}$ with $yBy^* > y\bar{B}_n y^* = yB_n(n)y^*$. On the other hand we must have $yB_n(n)y^* \geq yB(n)y^* = yBy^*$, a contradiction.

2.2 The importance of the matrix B lies in the following two lemmas. Let the matrix B' be defined with respect to the basis $x'_v = Tx_v$ ($v = 1, \dots, m$) of X' .

1) For a matrix C of arbitrary size over K we denote by C^* its conjugate transpose.

Lemma 2. In the situation of Lemma 1, assume that $n^-(V) = 0$ and $E = V \oplus X$. If the condition b) is replaced by

$$c) \quad B = B'$$

then the conclusion of Lemma 1 holds.

To formulate the next lemma we have to introduce some more notation. Let $0 \neq v \in V$ and $v' \in V'$ such that $\Phi'(v', Tx) = \Phi(v, x)$ for all $x \in X$ and $\Phi'(v', v') = \Phi(v, v)$, and set

$$\hat{V} := V \cap v^\perp, \quad \hat{E} := \hat{V} \oplus X.$$

Let $\hat{E} = (\hat{E}, \Phi, \hat{V})$ and define $\hat{E}'$ in a similar way. Finally let $\hat{B}, \hat{B}'$ be the associated matrices (with respect to the same bases of X, X').

Lemma 3. If we pass from E, E' to $\hat{E}, \hat{E}'$ condition c) of Lemma 2 is preserved, i.e. we have $\hat{B} = \hat{B}'$.

Proof of Lemma 2. We may assume that $\Phi(v, v) = 1$. It will be sufficient to find a vector $v' \in V'$ with

$$(3) \quad \Phi'(v', x'_v) = \gamma_v := \Phi(v, x_v) \quad \text{for } v = 1, \dots, m$$

and $\Phi'(v', v') < 1$, for the linear variety in V' cut out by (3) will then certainly contain vectors of length 1.

Now if n is so large that A'_n is invertible there are vectors $v' \in V'_n = K v'_1 \oplus \dots \oplus K v'_n$ with (3), and we would like to calculate the "length" $\Phi'(v', v')$ of the shortest of these vectors v' (n being fixed). To this end, let us identify K^m with X' by $(\lambda_1 \dots \lambda_m) \mapsto \sum \lambda_v x'_v$ so that we may interpret the matrix A'_n as a form on X' . After doing this we have an isometric injection $\varphi: (X', A'_n) \rightarrow (V'_n, \Phi')$, $x' \mapsto \varphi x' = \sum_{i=1}^n \Phi'(x', v'_i) v'_i$. Since for $v' \in V'_n$, $x' \in X'$ we have $\Phi'(v', x') = \Phi'(v', \varphi x')$ it is obvious that the shortest $v' \in V'_n$ with (3) lies in $\varphi X'$, $v' = \varphi x'$ with a uniquely determined $x' \in X'$. Condition (3)

translates to

$$x' A'_n = (\gamma_1 \dots \gamma_m) =: y \in K^n$$

so that $x' = y B'_n$. Hence $\Phi'(v', v') = x' A'_n x'^* = y B'_n y^*$. Is this < 1 for suitable n ? Since $B' = B$ it suffices to show that $y B y^* < 1$. Choose the orthonormal basis $(v_i)_i$ of V in such a way that $v_1 = v$. Then we find, for $\beta_n := y B_n y^*$, that $\beta_n = y B_n A_n B_n y^* = y B_n (A_1 + \hat{A}_{n-1}) B_n y^* = \beta_n^2 + y B_n \hat{A}_{n-1} B_n y^*$. Here $\hat{A}_{n-1} = A_n - A_1 = A_n - y^* y$ is associated to the embedding $\hat{E}$ by (1) and hence is positive definite for large n . We conclude that either $y = 0$ or else $\beta_n > \beta_n^2$ for large n . At any rate we get that $\beta_n < 1$ for large n . Because the β_n decrease monotonically towards $y B y^*$ we are done.

Proof of Lemma 3. We shall show that $\hat{B}$ depends only on B and y (keeping the notations of the foregoing proof), a fact which certainly implies Lemma 3. Indeed, from $\hat{B}_n = (A_{n+1} - y^* y)^{-1} = (1 - B_{n+1} y^* y)^{-1} B_{n+1}$ we conclude that $\hat{B} = \lim \hat{B}_n = (1 - B y^* y)^{-1} B$, if $(1 - B_{n+1} y^* y)^{-1}$ converges at all. But $(1 - B_{n+1} y^* y)^{-1} = \hat{B}_n A_{n+1} = \hat{B}_n (\hat{A}_n + y^* y) = 1 + \hat{B}_n y^* y$, and this converges towards $1 + \hat{B} y^* y$.

2.3 The last two lemmas can be reformulated by saying that T extends to arbitrary finite dimensional overspaces of X and X' and that these extensions form a set of partial isomorphisms (I.3.1). Hence T may be extended to an isometry $E \rightarrow E'$ mapping V onto V' . Let us record this fact explicitly as

Proposition 1. Let E, E' be 1-dense embeddings with $n^-(V) = n^-(V') = 0$, $\dim E/V = \dim E'/V' < \infty$, let $E = V \oplus X$, $E' = V' \oplus X'$ and let $T: X \rightarrow X'$ be an isometry. Then T admits an extension to an isometry $\bar{T}: E \rightarrow E'$ with $\bar{T}V = V'$ if and only if $B = B'$ (these matrices being defined with respect to any basis $x_1, \dots, x_m$ of X and its image basis $x'_v = T x_v$ in X').

This result is needed in order to establish the connection between the form $\Phi - \Phi_V = \Phi V \Phi$ on $C(V)$ and the matrix B (Corollary 3 below), which in turn is indispensable in the proof of Lemma 1.

Corollary 1. Let $X = X_1 \oplus \dots \oplus X_r$ such that each X_i is spanned by some of the x_v ($v = 1, \dots, m$). If B has a block diagonal shape with respect to this partitioning of $\{1, \dots, m\}$ then there is a decomposition $V = V_1 \oplus \dots \oplus V_r$ with each V_i of dimension N_0 , such that $V_i \perp X_j$ if $i \neq j$.

Proof. The idea is to compare the given embedding E with a suitably constructed embedding E' which enjoys the desired properties. Technically speaking, we let $E' = E$, $V' = V$, $X' = X$, $T: X \rightarrow X'$ the identity, $\Phi'|_{V \times V} = \Phi|_{V \times V}$, $\Phi'|_{X \times X} = \Phi|_{X \times X}$, but we redefine $\Phi|_{V \times X}$. Decompose $V = V_1 \oplus \dots \oplus V_r$, with each V_i of infinite dimension, and put $\Phi'|_{V_i \times X_j} = 0$ if $i \neq j$. It is explained in 2.4 below how to choose $\Phi'|_{V_i \times X_i}$ in such a way that $V \subset (E, \Phi')$ becomes 1-dense, and $B' = B$. Now the proposition applies.

Corollary 2. If B is diagonal then $X \cap C(V)$ is spanned by those x_v for which $B_{vv} \neq 0$.

Proof. According to Corollary 1 there is a decomposition $V = V_1 \oplus \dots \oplus V_m$ such that $V_v \perp x_\mu$ if $v \neq \mu$. If the orthonormal basis $(v_i)_{\mathbb{N}}$ of V is chosen in conformity with this decomposition then all the matrices A_n defined by (1) are diagonal and Corollary 2 becomes evident.

Note that there are always bases $\bar{x}_1, \dots, \bar{x}_m$ of X which render $\bar{B}$ diagonal. For, if $\bar{x}_v = \sum_{\mu} S_{v\mu} x_\mu$ is an arbitrary change of bases in X we have $\bar{A}_n = S A_n S^*$ and thus

$$(4) \quad \bar{B} = S^{-1*} B S^{-1}.$$

As announced earlier, the next corollary shows how the form $\Phi V \Phi$ and the matrix B are connected with each other. Although the statement appears quite natural, it is by no means obvious (unless $C(V) = E$), as is signalized by its failing in the case of other base fields in place of $\mathbb{R}, \mathbb{C}, \mathbb{H}$, e.g. $K = \mathbb{Q}$. (We shall inspect the possibility of transferring certain results to more general fields in a later chapter.)

Corollary 3. If the basis $x_1, \dots, x_m$ of X is chosen in such a way that $X \cap C(V)$ is spanned by $x_1, \dots, x_r$ then

$$B = \left(\begin{array}{c|c} C^{-1} & 0 \\ \hline 0 & 0 \end{array} \right)$$

where C is the matrix of $\Phi V \Phi$ with respect to $x_1, \dots, x_r$.

Proof. Let $\bar{x}_1, \dots, \bar{x}_m$ be a basis with $\bar{B}$ diagonal. By Corollary 2 we may assume that $X \cap C(V)$ is also spanned by $\bar{x}_1, \dots, \bar{x}_r$. Let $\bar{C}$ be the matrix of $\Phi V \Phi$ with respect to $\bar{x}_1, \dots, \bar{x}_r$. The transformation matrix S has the shape $S = \left(\begin{array}{c|c} R & 0 \\ \hline * & * \end{array} \right)$. By (4) we find

$$B = S^* \bar{B} S = \left(\begin{array}{c|c} R^* \bar{C}^{-1} R & 0 \\ \hline 0 & 0 \end{array} \right)$$

where $\bar{B} = \left(\begin{array}{c|c} \bar{C}^{-1} & 0 \\ \hline 0 & 0 \end{array} \right)$ (for the basis $\bar{x}_1, \dots, \bar{x}_m$, the assertion of Cor. 3 follows from the proof of Corollary 2). But $\bar{C} = R C R^*$, so that $R^* \bar{C}^{-1} R = C^{-1}$.

Now we are in a position to finish the proof of Lemma 1 (in the case $n(V) = 0$). Hereby we may assume that $X \cap V = 0$, as we observed at the beginning of 2.1. Choose a basis $x_1, \dots, x_m$ of X with the property indicated in Corollary 3. Since $T(X \cap C(V)) = X' \cap C(V')$ the image basis has the analogous property in E' . From Corollary 3 and condition b) in Lemma 1 we infer that $B = B'$. Now apply Lemma 2!

2.4 Here we would like to outline the construction of $\perp$ -dense embeddings E with $n^-(V) = 0$ and with prescribed (positive hermitean) matrix B and form $\Phi|_{X \times X}$, since this construction was used in the proof of Corollary 1 above. We begin by choosing an invertible matrix S such that $\bar{B} := S^{-1*} B S^{-1}$ turns out diagonal, say $\bar{B} = \text{diag}(\bar{b}_1, \dots, \bar{b}_m)$. Let $\bar{x}_v := \sum_{\mu} S_{v\mu} x_{\mu}$, where $x_1, \dots, x_m$ is the given basis of X , and define $\Phi|_{V \times V}$ positive definite and $\Phi|_{X \times X}$ as prescribed. To define $\Phi|_{V \times X}$, decompose $V = V_1 \dot{\oplus} \dots \dot{\oplus} V_m$, each V_{μ} infinite dimensional, and let $(v_{\mu i})_{i \in \mathbb{N}}$ be an orthonormal basis of V_{μ} ($\mu = 1, \dots, m$). Then put

$$\begin{aligned} \Phi(v_{\mu}, \bar{x}_v) &= 0 & \text{if } v \neq \mu \\ \Phi(v_{\mu i}, \bar{x}_{\mu}) &= \alpha_{\mu i} \end{aligned}$$

where the $\alpha_{\mu i}$ are nonzero real numbers such that

$$\sum_{i \in \mathbb{N}} \alpha_{\mu i}^2 = \bar{b}_{\mu}^{-1} \quad (= \infty \quad \text{if } \bar{b}_{\mu} = 0).$$

It is easy to verify that $V^{\perp} = 0$, and it is clear that the matrix of E defined by (2) with respect to $\bar{x}_1, \dots, \bar{x}_m$ is $\bar{B}$. Hence by (4) its matrix relative to $x_1, \dots, x_m$ is B .

3. Proof of Lemma 1 for arbitrary $n^-(V) < \infty$

We shall reduce the problem through several steps to the case $n^-(V) = 0$ already done.

3.1 Chopping off nondegenerate $Y \subset X \cap C(V)$. This first reduction is similar to that at the beginning of 2.1. If Y is a nondegenerate subspace of $X \cap V$, decompose $V = Y \dot{\oplus} V_0$, $v = y + v_0$, $X = Y \dot{\oplus} X_0$ with $X_0 \supset X \cap V_0 = X \cap V \cap Y^{\perp}$ (it would be simpler to require $X_0 \perp Y$, but we prefer to maintain the possibility of keeping inside X_0 a fixed supplement of V in E during all reduction steps of this kind), and put $E_0 = V_0 + X_0$. Let $Y' = TY$, $X_0' = TX_0$ and define V_0', E_0' in a

similar way. Now, replacing E, V, T, v etc. by $E_0, V_0, T|_{X_0}, v_0$ etc. is legal since the assumptions of Lemma 1 are easily seen to go through.

3.2 How to get rid of $X \cap V$. Thanks to 3.1 we may assume that $X \cap V$ is totally isotropic. Pick $z \in X \cap V$, $z \neq 0$, and set $X = Kz \oplus X_1$. Then $X_1^\perp \cap V$ is not contained in $z^\perp \cap V$; otherwise we would have $X_1 \supset Kz$, since $(X_1^\perp \cap V)^\perp = X_1$ by the 1-density of V . Choose $y \in X_1^\perp \cap V$ with $\phi(y, z) = 1$. Similarly there exists $y' \in (TX_1)^\perp \cap V'$ with $\phi'(y', Tz) = 1$, and we may even arrange for $\phi(y, y) = \phi'(y', y')$ by correcting y or y' (depending on whether $\phi(y, y) > \phi'(y', y')$ or $\phi(y, y) < \phi'(y', y')$) modulo $V \cap X^\perp \cap y^\perp$ or $V' \cap X'^\perp \cap y'^\perp$. Now extend T onto $X \oplus Ky$ by $y \mapsto y'$ and chop off the hyperbolic planes $Y = Kz \oplus Ky$, $Y' = KTz \oplus Ky'$, using 3.1. After a finite number of such steps we have $X \cap V = 0$.

3.3 Now that $X \cap V = 0$ the proof of Lemma 1 will be given by induction upon $m = \dim X$, the case $m = 0$ being trivial. The idea is to set up decompositions (possibly after some further reduction steps)

$$\begin{aligned}
 (5) \quad X &= Kx \oplus X_1 \\
 V &= V_- \overset{1}{\oplus} V_+ , \quad V_+ = V_1 \overset{1}{\oplus} V_2 \\
 V' &= V'_- \overset{1}{\oplus} V'_+ , \quad V'_+ = V'_1 \overset{1}{\oplus} V'_2 \\
 &\text{such that}
 \end{aligned}$$

$$\alpha) \quad x \perp V_- + V_1 , \quad Tx \perp V'_- + V'_1$$

$$\beta) \quad X_1 \perp V_2 , \quad TX_1 \perp V'_2 .$$

Suppose that we are able to find such decompositions. Consider the embeddings $V_- + V_1 \subset V_- + V_1 + X_1$, $V'_- + V'_1 \subset V'_- + V'_1 + TX_1$ and $T_1 := T|_{X_1}$. From β) we conclude that they are 1-dense and that the assumptions of Lemma 1 are satisfied. Hence, by induction assumption, we may extend T_1 onto $X_1 + V_-$, with $T_1 V_- \subset V'_- + V'_1$. But by α) this extension is

compatible with $x \mapsto Tx$ whence we have in fact extended T onto $X+V_-$. If we now chop off V_- and TV_- according to 3.1 we find ourselves in the case $n(V) = 0$.

3.4 The case $C(V) \neq E$. Choose a basis $x_1, \dots, x_m$ of X such that $X \cap C(V)$ is spanned by $x_1, \dots, x_r$, and let $x := x_m$, $X_1 := Kx_1 + \dots + Kx_{m-1}$. We shall first establish a fundamental decomposition $V = V_- \oplus V_+$ with $V_- \perp x$. To this end, start with an arbitrary fundamental decomposition $V = V_- \oplus V_+$. The $\perp$ -dense embedding $V_+ \subset V_+ \oplus Kx$ has $B = 0$; hence, by using Proposition 1 and 2.4 we see that there exists an orthonormal basis $(v_i)_{i \in \mathbb{N}}$ of V_+ such that $\phi(v_i, x) = i$ for all $i \in \mathbb{N}$. Let $w_1, \dots, w_n$ be a (-1) -orthogonal basis of V_- and choose $i_0 > \max\{|\phi(w_1, x)| \mid i = 1, \dots, n\}$. Let $\lambda_1 \in K$ such that for $\bar{w}_i := w_i + \lambda_1 v_{i_0+i}$ we have $\phi(\bar{w}_i, x) = 0$ ($i = 1, \dots, n$). Then $|\lambda_1| < 1$ and thus $\phi(\bar{w}_i, \bar{w}_i) < 0$. Since the $\bar{w}_i$ are also orthogonal to each other we may replace V_- by $K\bar{w}_1 \oplus \dots \oplus K\bar{w}_n \perp x$ and V_+ by its orthogonal complement in V . To arrive at (5) there remains to decompose V_+ as $V_1 \oplus V_2$ such that $V_1 \perp x$, $V_2 \perp X_1$. This is achieved by applying Corollary 1 of Proposition 1 to the embedding $V_+ \subset V_+ \oplus X$. The block diagonal shape of B_+ required to this is warranted by Corollary 3 of Proposition 1 and by the choice of $x_1, \dots, x_r$ (note that $C(V_+) \cap X = C(V) \cap X$). The required decomposition of V' is gained in the same manner.

3.5 The case $C(V) = E$. Let us denote by χ the restriction to X of the form $\phi - \phi_V = \phi V \phi$. Every fundamental decomposition of $V = V_- \oplus V_+$ gives rise to a decomposition $\chi = \chi_+ + \chi_-$ with χ_+ positive definite and χ_- negative, namely $\chi_+ = (\phi V_+ \phi)|_{X \times X}$, $\chi_- = (\phi V_- \phi)|_{X \times X}$. Assume that χ_- is definite for some fundamental decomposition of V . This means that the functionals $V_- \rightarrow K$, $v \mapsto \phi(v, x_v)$, $v = 1, \dots, m$

(where $x_1, \dots, x_m$ is a basis of X) are linearly independent. Hence there is $w \in V_-$ with arbitrarily prescribed $\phi(w, x_v)$. We choose $\phi(w, x_v) = \phi'(w', Tx_v)$ where $w' \in V'$ is any vector with $\phi'(w', w') < 0$ and $\phi'(w', Tx_1) \neq 0$. Then $w \neq 0$ and we may arrange for $\phi(w, w) = \phi'(w', w') < 0$. Now we extend T by $w \mapsto w'$ and split off the lines Kw, Kw' according to 3.1. After having repeated this procedure as often as possible we have that χ_- and χ'_- turn out degenerate for every choice of the fundamental decompositions of V and V' . If $\dim X = 1$ this implies that $n^-(V) = 0$, so this case is done. If $\dim X > 1$ we would like to find $x \in X$ such that both $\chi_-(x, x) = 0$ and $\chi'_-(Tx, Tx) = 0$ for suitable fundamental decompositions. Let $V = V_- \overset{1}{\oplus} V_+$ and choose nonzero $x \in X$ with $\chi_-(x, x) = 0$, and consider the embeddings $V \subset V \oplus Kx$, $V' \subset V' \oplus KTx$. By the $m = 1$ case just done we may extend $T|_{Kx}$ onto $V_- \oplus Kx$ and decompose $V' = V'_- \overset{1}{\oplus} V'_+$, with V'_- the image of V_- . For this decomposition we have $\chi'_-(Tx, Tx) = 0$. If we put $X = Kx \oplus X_1$ with X_1 the orthogonal complement of Kx with respect to χ_+ , we are in a position to apply Corollary 1 of Proposition 1 to the embedding $V_+ \subset V_+ \oplus X$. The issue is a decomposition $V_+ = V_1 \overset{1}{\oplus} V_2$ with $V_1 \perp x$ and $V_2 \perp X_1$, as required in (5) ($x \perp V_-$ follows from $\chi_-(x, x) = 0$). To get a similar decomposition of V'_+ we should have that Tx and TX_1 are orthogonal to each other with respect to χ'_+ . But $x \in \text{rad } \chi_-$ and $x' \in \text{rad } \chi'_-$ so that orthogonality with respect to χ_+, χ'_+ is tantamount to orthogonality with respect to χ, χ' . Since T transforms χ into χ' we are done.

The proof of Lemma 1 is thus complete.

CHAPTER III

THE LATTICE

In the present chapter we shall attach a lattice $V(E)$ to every embedding E , and we shall gather all the information on $V(E)$ which is needed later on. This lattice plays a crucial part inasmuch as it enables us to define invariants which characterize E up to congruence.

1. Definition of $V(E)$

Let $E = (E, \phi, V)$ be an arbitrary embedding. The main new feature (compared with the dense case) consists, of course, in the presence of the space $V^\perp$. Thus, we should also pay attention to the ghost form of $V^\perp$ and, instead of considering the mere space $(C(V), \phi_V)/r\phi_V$ we should consider the embedding $(C(V), \phi_V, V^\perp + r\phi_V)/r\phi_V$. The procedure calls for repetition, so let us, by recursion, define sequences $(E_i)_{i \in \mathbb{N}}$, $(V_i)_{i \in \mathbb{N}}$ of subspaces of E , and on each E_i a form ϕ_i , as follows.

$$(1) \quad \begin{aligned} E_0 &= E, & \phi_0 &= \phi, & V_0 &= V \\ E_{i+1} &= C(E_i, \phi_i, V_i), & \phi_{i+1} &= (\phi_i)_{V_i}, & V_{i+1} &= r\phi_{i+1} + V_i^{\perp i} \end{aligned}$$

The symbol $\perp_i$ means orthocomplementation in (E_i, ϕ_i) . Furthermore we set

$$(2) \quad V_\infty := \bigcup_{i \in \mathbb{N}} V_i, \quad E_\infty := \bigcap_{i \in \mathbb{N}} E_i$$

so that $V_\infty \subset E_\infty$, and finally we put

$$(3) \quad F_i := C(E_i, \phi_i, V_i^{\perp i}), \quad \psi_i := (\phi_i)_{V_i^{\perp i}}.$$

Thus ψ_i is a form on F_i , with radical $r\psi_i = V_i^{\perp i}$, by (2) in I.2.

The lattice $V(E)$ is now defined to be the smallest sublattice of $L(E)$, the full lattice of subspaces of E , which contains all the

spaces $E_i, F_i, V_i, E_\infty, V_\infty$ introduced in (1), (2), (3), and which is $\perp$ -stable with respect to each of the forms ϕ_i, ψ_i . (Actually it would suffice to require $\perp$ -stability with respect to each ϕ_i , as we shall see.) By $\perp$ -stability with respect to ϕ_i we mean

$$A \in V(E), A \subset E_i \Rightarrow A^{\perp i} \in V(E).$$

We start our investigation of $V(E)$ by listing certain relations among its elements. We shall use the abbreviations

$$R_i := V_i \cap V_i^{\perp i}, \quad S_i := V_i^{\perp i} \cap V_i^{\perp i \perp i}.$$

In all these sequences of spaces and forms, the subscript $i=0$ shall frequently be dropped.

Lemma 1. For all $i \geq 0$ we have

- a) $V_{i+1}^{\perp i+1} = E_{i+1} \cap V_i^{\perp i \perp i}$
- b) $R_{i+1} = r\phi_{i+1} + S_i$
- c) $R_{i+1}^{\perp i+1} = E_{i+1} \cap S_i^{\perp i}$
- d) $E_{i+1}^{\perp i} = r\phi_{i+1} \cap V_i^{\perp i}$
- e) $E_{i+2} = E_{i+1} \cap F_i$ and $\phi_{i+2} = \phi_{i+1} - \phi_i + \psi_i$
(restricting all forms to E_{i+2}).

Proof. Every space occurring in the lemma lies between $r\phi_i$ and E_i , hence it suffices to consider the typical case $i=0$.

- a) $V_1^{\perp 1} = (r\phi_1 + V^{\perp})^{\perp 1} = (V^{\perp})^{\perp 1} = E_1 \cap V^{\perp \perp}$ by I.2 (1).
- b) $R_1 = V_1 \cap V_1^{\perp 1} = V_1 \cap V^{\perp \perp}$ by a), hence $R_1 = (r\phi_1 + V^{\perp}) \cap V^{\perp \perp} = r\phi_1 + S$, since $r\phi_1 \subset V^{\perp \perp}$.
- c) From b) it follows that $R_1^{\perp 1} = S^{\perp 1}$, hence $R_1^{\perp 1} = E_1 \cap S^{\perp 1}$ by I.2 (1).
- d) Since $V \subset E_1$ we have $E_1^{\perp} \subset V^{\perp}$, hence from I.2 (1) we conclude

that $E_1^\perp = E_1^\perp \cap V^\perp = E_1^{\perp 1} \cap V^\perp = r\phi_1 \cap V^\perp$.

e) Assume first that both $n^+(V_1^\perp, \phi) = n^+(V_1, \phi_1)$ and $n^-(V_1^\perp, \phi) = n^-(V_1, \phi_1)$ are infinite. By definition, this implies $F = V^\perp + V^{\perp 1}$ and $E_2 = V_1 + V_1^{\perp 1}$. Hence, by a), $E_2 = V_1 + E_1 \cap V^{\perp 1} = V^\perp + E_1 \cap V^{\perp 1} = E_1 \cap (V^\perp + V^{\perp 1}) = E_1 \cap F$. Let then, say, $n^-(V^\perp) < \infty$. We have $F \cap E_1 = F \cap S^\perp \cap E_1 = F \cap R_1^{\perp 1}$, by c). From this and $\phi|_{V^\perp \times E_1} = \phi_1|_{V^\perp \times E_1}$ we infer that $F \cap E_1 = E_2$. In both cases, $\phi_2(x, y) = \phi_1(x, y) - (\phi_1 V_1 \phi_1)(x, y) = \phi_1(x, y) - \phi(x, y) + \phi(x, y) - (\phi V^\perp \phi)(x, y) = \phi_1(x, y) - \phi(x, y) + \psi_0(x, y) \quad (x, y \in E_2)$.

2. The lattices $V_i(E)$

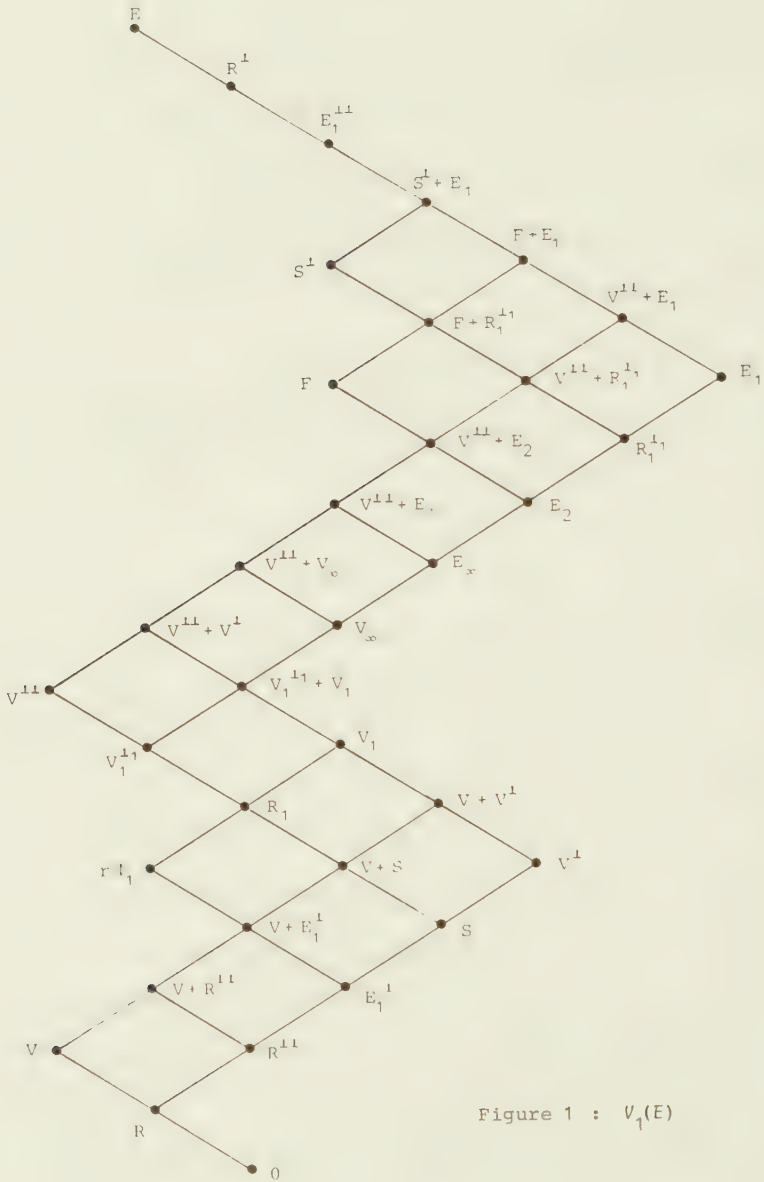
2.1 In order to gain insight into the structure of $V(E)$ we introduce the sublattices $V_i(E)$ (for $i \geq 1$), which are defined to be the smallest sublattices of $V(E)$ satisfying the conditions

- (4) (i) $E_\infty, V_\infty \in V_1(E)$
 (ii) if $j \leq i$ then $E_j, V_j, r\phi_j \in V_i(E)$
 (iii) if $j < i$ then $F_j \in V_i(E)$
 (iv) if $j < i$ then $V_i(E)$ is $\mathfrak{u}$ -stable with respect to ϕ_j and ψ_j

Thus we have $V(E) = \bigcup_{i \in \mathbb{N}} V_i(E)$. We shall give a recursive construction of the $V_i(E)$, and we begin by computing the lattice $V_1(E)$.

Lemma 2. The lattice $V_1(E)$ is given by the diagram in Figure 1.

Proof. The reader is invited to consult [12], p. 172 as far as the general procedure for the verification of diagrams is concerned. It is easy to check that the mapping from the unlabelled diagram (which is, of course, a lattice) to $L(E)$ induced by the labelling is a lattice homomorphism. All the nontrivial relations needed for that follow from Lemma 1 (e.g., $E_1^\perp = r\phi_1 \cap V^\perp$). As to the $\mathfrak{u}$ -stability with respect to ϕ , we note that, e.g., every space Y between $V + V^\perp$ and $S^\perp$ has $Y^\perp = S$.

Figure 1 : $V_1(E)$

Finally, the $\perp$ -stability with respect to $\Psi_0 = \Psi$ follows from the fact that the smallest space in the interval $[V^\perp = r\Psi, F]$ different from $r\Psi$ (namely, $V + V^\perp$) is already $\perp$ -dense: $(V + V^\perp)^\perp \Psi = V^\perp \Psi = V^\perp$, by 1.2(1).

2.2 The embedding E gives rise to further embeddings,

$$(5) \quad E_i := (E_i, \phi_i, V_i) / r\phi_i \quad (i \in \mathbb{N}).$$

It is useful to think of $L(E_i / r\phi_i)$ as a sublattice of $L(E)$, by identifying each $Y \subset E_i / r\phi_i$ with its inverse image under the canonical projection $E_i \longrightarrow E_i / r\phi_i$. Using this convention we have

Theorem 1. For each $n \geq 1$ the lattice $V_{n+1}(E)$ is finite and distributive. It is additively generated by $V_1(E)$ and $V_n(E_1)$; more exactly, $V_{n+1}(E) = V_1(E) \cup V_n(E_1) \cup (V^{\perp\perp} + V_n(E_1)) \cup (F + V_n(E_1))$, and $F + V_n(E_1)$ is already contained in $V_2(E)$.

Before entering the proof of this theorem we insert a lattice theoretical lemma.

Lemma 3. Let L be a modular lattice and $M \subset L$ a sublattice which has a smallest element x_0 and a greatest element x_1 and which is generated by two chains C_1, C_2 with $x_0, x_1 \in C_1 \cap C_2$. Let $a \in L$ be such that $a \wedge x_1 \in C_1$ and put $C_{11} := [x_0, a \wedge x_1] \subset C_1$, $C_{12} := [a \wedge x_1, x_1] \subset C_1$ (intervals taken in C_1). Then the lattice $\langle M, a \rangle$ generated by M and a equals $M \cup (a \vee M)$ and is generated by the two chains $C'_1 = C_{11} \cup (a \vee C_{12})$, C_2 .

Proof. If $x \in C_{12}$ the relation $(a \vee x) \wedge x_1 = (a \wedge x_1) \vee x = x$ shows that $x \in \langle C'_1, C_2 \rangle$. Hence $\langle M, a \rangle = \langle C_1, C_2, a \rangle$ equals $\langle C'_1, C_2 \rangle$ and is therefore distributive [13]. From this we conclude easily that $M \cup (a \vee M)$ is a sublattice of L , whence $M \cup (a \vee M) = \langle M, a \rangle$.

Proof of Theorem 1. Let us show, by induction on n , the three following assertions:

- (6) $V_n(E)$ is generated by two chains C_1, C_2 with $V^{\perp}, E_1 \in C_1$ and $V \in C_2$
- (7) $V_n(E) = [0, V^{\perp}] \cup [V, E]$
- (8) The intervals $[R_1^{\perp}, E]$ and $[0, V + V^{\perp}]$ of $V_n(E)$ are already contained in $V_1(E)$.

The distributivity of $V_n(E)$ then follows from (6), and the remaining assertions of the theorem will result from the proof of (6), (7) and (8). A glance at the diagram of $V_1(E)$ (Figure 1) reveals that (6) and (7) hold for $n=1$, and (8) is trivial. Taking for granted the assertions up to a certain n we are to prove them for $n+1$. By induction assumption, the lattice $\omega := V_n(E_1) \subset V(E)$, which has smallest element $r\phi_1$ and greatest element E_1 , is generated by two chains K_1, K_2 with $V_1^{\perp}, E_2 \in K_1$ and $V_1 \in K_2$. We may assume that $r\phi_1, E_1 \in K_1 \cap K_2$. We begin by computing the lattice $\langle V_1(E), \omega \rangle$ by iterated application of Lemma 3. Indeed, since $V^{\perp} \cap E_1 = V_1^{\perp}$ (Lemma 1) belongs to K_1 , the lattice $\langle V^{\perp}, \omega \rangle = \omega \cup (V^{\perp} + \omega) =: \omega'$ is generated by the two chains $K'_1 = \{X \in K_1 \mid X \subset V_1^{\perp}\} \cup \{V^{\perp} + X \mid X \in K_1, V_1^{\perp} \subset X\}$, $K'_2 = K_2 \cup \{V^{\perp} + E_1\}$, and since $F \cap (V^{\perp} + E_1) = V^{\perp} + E_2$ belongs to K'_1 the lattice $\langle F, \omega' \rangle = \omega' \cup (F + \omega') =: \omega''$ is generated by two chains K''_1, K''_2 with

(9) $V_1, E_1 \in K''_2$; $V_1^{\perp}, F \in K''_1$.

Now, are there elements of $V_1(E)$ missing in ω'' ? Looking again at the diagram of $V_1(E)$ we find that such is the case for the intervals $[0, V + V^{\perp}]$ and $[S^{\perp}, E]$ of $V_1(E)$. We contend that $\langle V_1(E), \omega \rangle = \omega'' \cup [0, V + V^{\perp}] \cup [S^{\perp}, E]$. To prove this, note that by (8) applied to $\omega = V_n(E_1)$, the intervals $[R_1^{\perp}, E_1], [r\phi_1, V_1]$ of ω are the sets

$\{R_1^{\perp 1}, E_1\}$, $\{r\phi_1, R_1, V_1\}$, respectively, and thus lie in $V_1(E)$. The same holds for the interval $[F + R_1^{\perp 1}, F + E_1] = \{F + R_1^{\perp 1}, F + E_1\}$ of ω . Hence our claim can be read off at once from Figure 1. Next we extend the chains K_2'', K_1'' in $\langle V_1(E), \omega \rangle$ in the following way. Put $C_1 := \{0, R_1^{\perp 1}, V_1^{\perp}\} \cup K_2'' \cup \{E_1^{\perp 1}, R_1^{\perp}, E\}$, $C_2 := \{0, V\} \cup (K_1'' \setminus \{F + E_1\}) \cup \{S^{\perp}, E\}$, which by (9) are chains. We see that $\langle V_1(E), \omega \rangle$ is generated by C_1, C_2 , with $V_1^{\perp}, E_1 \in C_1$ and $V \in C_2$ so that (6) will be satisfied for $V_{n+1}(E)$ if we can show that $V_{n+1}(E) = \langle V_1(E), \omega \rangle$. The inclusion $\langle V_1(E), \omega \rangle \subset V_{n+1}(E)$ is obvious, and for the reverse inclusion we have to check the conditions (4) with $i = n+1$. Among these only (iv) is nontrivial. Because $\omega = V_n(E_1)$ is $\perp$ -stable with respect to each ϕ_j, ψ_j if $1 \leq j \leq n$ and since the interval $[r\phi_1, E_1]$ of $\langle V_1(E), \omega \rangle$ equals ω (by construction) we only have to check $\perp$ -stability with respect to ϕ and ψ . As to ϕ , it is enough to test the orthogonals of those $+$ -irreducible elements which are not in $V_1(E)$. By the construction of $\langle V_1(E), \omega \rangle$ all these elements lie in ω ; more exactly, they lie in one of the intervals $[V_1, R_1^{\perp 1}]$, $[R_1, V_1^{\perp 1}]$ (for, $\omega = [V_1, R_1^{\perp 1}] \cup [R_1, V_1^{\perp 1}] \cup [r\phi_1, E_1]$ by (7) and (8)). But each $Y \in [V_1, R_1^{\perp 1}]$ has $Y^{\perp} = S$ and each $Y \in [R_1, V_1^{\perp 1}]$ has $Y^{\perp} = V^{\perp}$. As to ψ , we observe that each element of $\langle V_1(E), \omega \rangle$ not in $V_1(E)$ contains V , and $V^{\perp\psi} = V^{\perp} = r\psi$, as noted in the proof of Lemma 2. Hence we have shown that $V_{n+1}(E) = \langle V_1(E), \omega \rangle$. Condition (6) has already been settled, and (7), (8) are evident from the construction of $V_{n+1}(E)$.

All the assertions of Theorem 1, with the exception of the last, are clear from what we have done so far. To show that $F + V_n(E_1) \subset V_2(E)$ is tantamount to showing that $F + V_n(E_1) \subset F + V_1(E_1)$. Now, $F + V_n(E_1) = F + [F \cap E_1, E_1] = F + [E_2, E_1] \subset F + [R_2^{\perp 2}, E_1] \subset F + V_1(E_1)$, where the last inclusion is obtained by applying (8) to $V_n(E_1)$.

2.3 As an illustration to Theorem 1 we have plotted the diagrams of $V_2(E)$ and $V_3(E)$ (Figures 2 and 3). They will also be helpful as a guide of orientation in the proof of the Congruence Theorem in the next chapter. Let us note some consequences of Theorem 1. Since $V(E) = \bigcup V_i(E)$ we have

Corollary 1. The lattice $V(E)$ is distributive.

A situation of special interest arises if we assume that for all $i \in \mathbb{N}$ we have $V_i^{i+1,i} \subset E_{i+1}$ (equivalently, $V_i^{i+1,i} = V_{i+1}^{i+1}$). In this case the lattice $V(E)$ takes on the particularly simple form indicated in Figure 4. In the general case the diagram in Figure 4 still represents a sublattice $V_{\text{red}}(E)$ of $V(E)$, and we have¹⁾

Corollary 2. The lattice $V(E)$ is additively generated by $V_{\text{red}}(E)$ and the set $\{V_i^{i+1,i} \mid i \in \mathbb{N}\}$.

This last result gives a reasonably clear picture of $V(E)$, provided that there is no collapsing in the lattice, i.e. the canonical mapping from the free lattice V (to be introduced in the next section) to $V(E)$ is injective. (We shall see later that such general embeddings exist.) Indeed, most of the properties of $V(E)$ proved in the next section could be read off directly from the diagrams if there is no collapsing.

-
- 1) It is easy to give a formal definition of $V_{\text{red}}(E)$: First, let $V_{1,\text{red}}(E) := [0, E_1] \cup [F, E]$ (intervals of $V_1(E)$), then put $V_{n+1,\text{red}}(E) := V_{1,\text{red}}(E) \cup V_{n,\text{red}}(E_1) \cup (F + V_{n,\text{red}}(E_1))$ (which is a sublattice of $V_{n+1}(E)$, as follows from the proof of Theorem 1), and finally let $V_{\text{red}}(E) := \bigcup_{n \in \mathbb{N}} V_{n,\text{red}}(E)$.

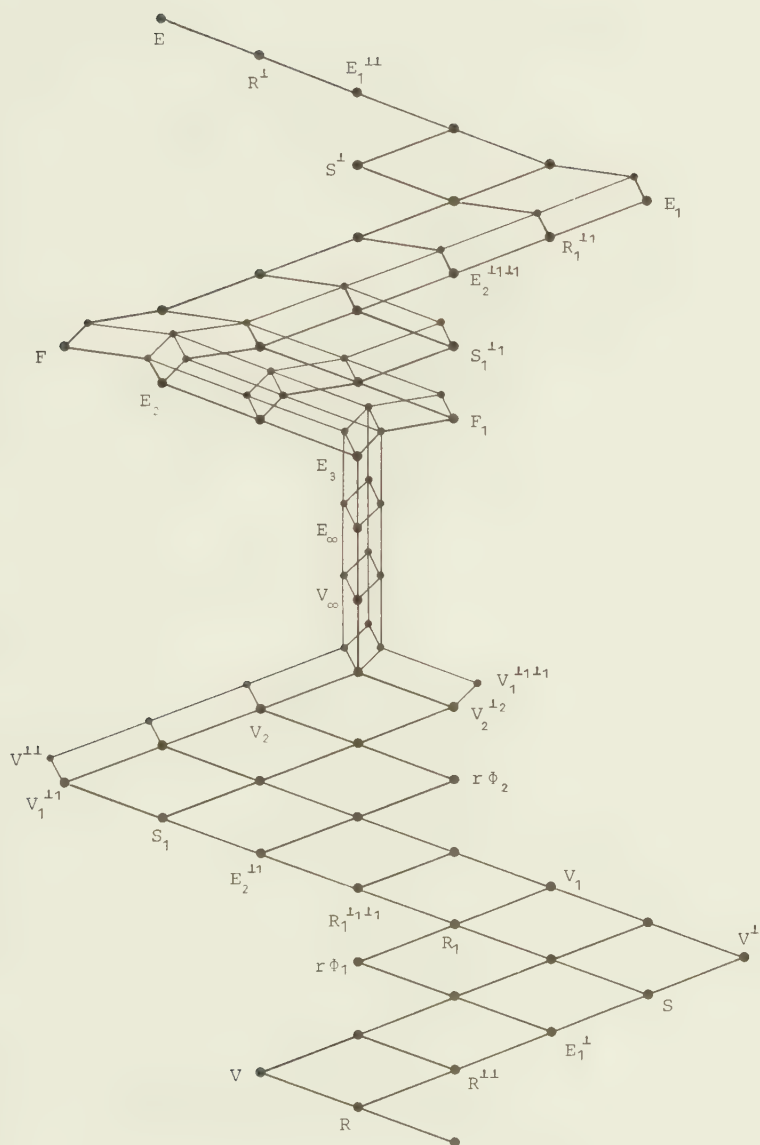
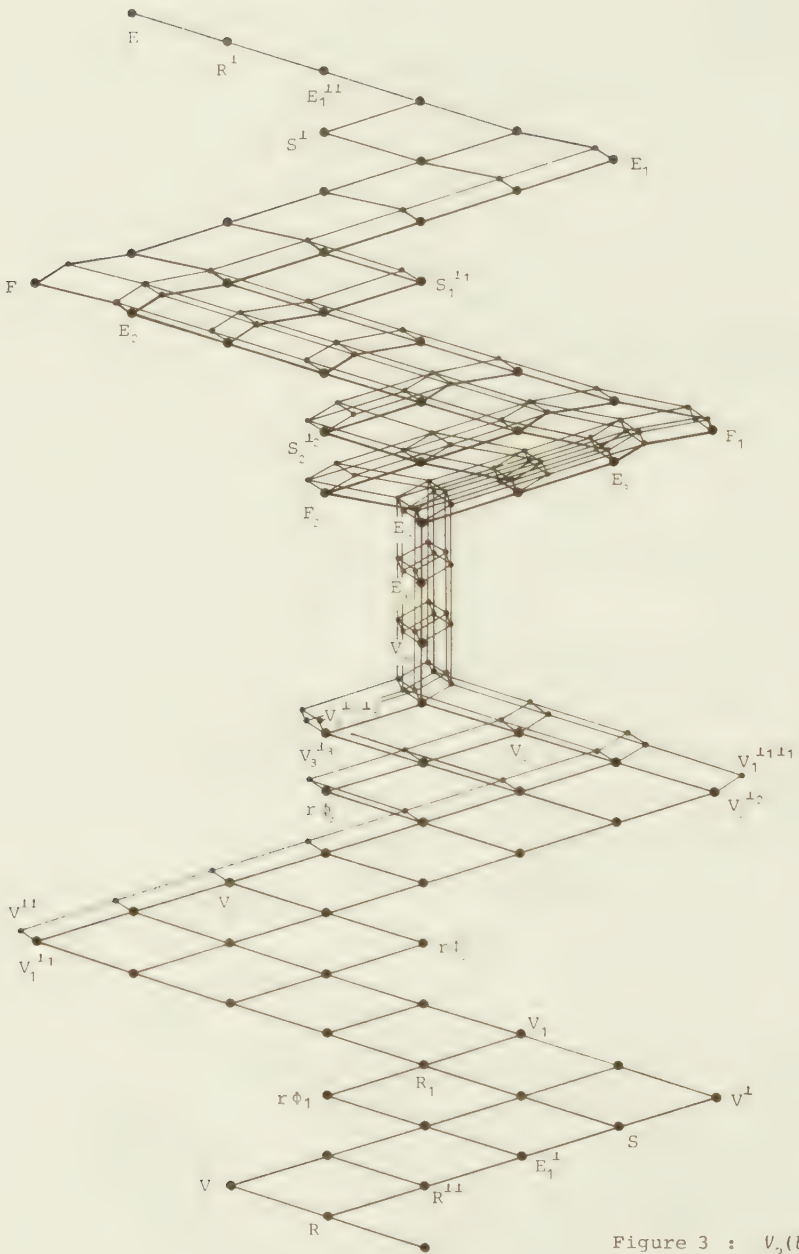


Figure 2 : $V_2(E)$

Figure 3 : $V_3(E)$

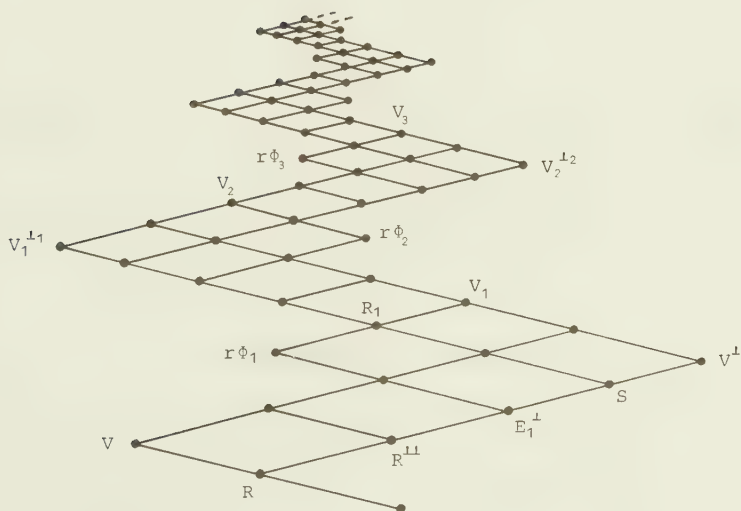
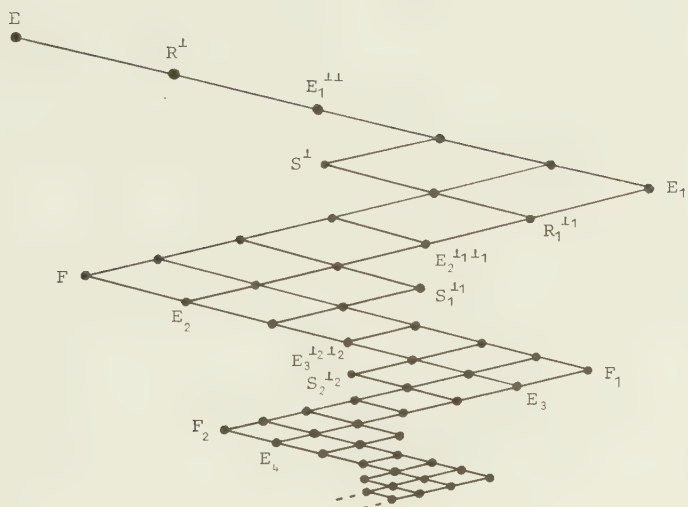


Figure 4 : $\mathcal{V}_{\text{red}}(E)$

3. The free lattice

3.1 Consider, e.g., the diagram of $V_3(E)$, Figure 3. This diagram can be interpreted as an abstract lattice, call it V_3 , to strictly distinguish it from the concrete sublattice $V_3(E) \subset L(E)$. Given any embedding E , the labelling of V_3 induces a canonical lattice homomorphism from V_3 onto $V_3(E)$. In the same fashion we get, for each $n \geq 1$, abstract lattices V_n ; the proof of Theorem 1 shows precisely how V_{n+1} is to be constructed if V_n is already known (instead of considering $V_n(E_1)$ one has to raise the subscripts of the elements of V_n by one). The inclusions $V_n(E) \subset V_{n+1}(E)$ are replaced by canonical injections $V_n \hookrightarrow V_{n+1}$, so that we may form the direct limit

$$V := \varinjlim_{n \rightarrow \infty} V_n$$

which is the free lattice attached to our Witt problem. As a matter of notation we shall denote the elements of V by small letters, e.g., e_i , e_i^{1i} , f_i , v_i^{1i1i} , etc. Thus the canonical lattice homomorphism

$$v : V \longrightarrow V(E)$$

is given by $v(e_i) = E_i$, $v(e_i^{1i}) = E_i^{1i} = r\phi_i$, and so on. The lattice operations in V shall also be written as $\cap$ and $+$. The assertions in 2.3 above are, actually, statements concerning the free lattice V . Thus we retain that V is distributive and is additively generated by the sublattice V_{red} and the subset $\{v_i^{1i1i} \mid i \in \mathbb{N}\} \subset V$.

When one deals with problems which involve finite lattices only, there is usually no need to make an explicit distinction between free and concrete lattices. In the case of infinite lattices, however, one has to grapple with problems which are a matter of course in the former case. E.g., does every $+-$ irreducible element $A \in V(E)$ come from a $+-$ irre-

ducible element of V ? Is there a smallest element $a \in V$ with $v(a) = A$? Questions of this kind shall be treated in the sequel.

We begin by putting down some more or less obvious facts concerning the free lattice alone. We use the abbreviations

$$(10) \quad \begin{aligned} z_n &:= v_0^{1_0 1_0} + \dots + v_{n-1}^{1_{n-1} 1_{n-1}} \quad (n \geq 1) \\ z_n &= v(z_n) \in V(E), \quad Z = \bigcup_{n \geq 1} z_n = \sum_{i \in \mathbb{N}} v_i^{1_i 1_i}. \end{aligned}$$

Notice that the space $Z \subset E$ does not in general belong to $V(E)$, but $E_n + Z = E_n + z_n$ does.

Proposition 1.

- a) The intervals $[0, z_n], [e_n, e]$ of V are finite; in the interval $[0, e_\infty + z_n]$ the minimum condition holds.
- b) If $(a_i)_{i \in \mathbb{N}}$ is a strictly decreasing sequence in V then for every $n \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that $a_i \leq e_n + z_n$. In particular,
- $$\bigcap_{i \in \mathbb{N}} v(a_i) \subset \bigcap_{n \in \mathbb{N}} (E_n + Z).$$
- c) If $a \not\leq e_n + z_n$ then $a \geq e_i$ for some i .

Proof a). Each $v_i^{1_i 1_i}$ has a unique immediate predecessor in V , namely, $v_{i+1}^{1_{i+1} 1_{i+1}}$. Hence (Figure 4) $[0, v_i^{1_i 1_i}]$ is finite, and by distributivity $[0, z_n]$ is finite as well. The finiteness of $[e_n, e]$ is obtained by an easy induction argument. The interval $[0, e_\infty]$ of V coincides with the corresponding interval of V_{red} , in which the minimum condition obviously holds (Figure 4). Using again the distributivity of V , we have the minimum condition in $[0, e_\infty + z_n]$.

b). This is proved by induction on n . For $n = 1$, note that $V \setminus [0, e_1 + z_1] = [f, e]$ is finite. Let n be arbitrary and assume that we have already found $i \in \mathbb{N}$ with $a_i \leq e_n + z_n$. Since $[0, z_n]$ is finite and since for $j \geq i$ we have the relation $a_j = e_n \cap a_j + z_n \cap a_j$ the

sequence $(e_n \cap a_j)_{j \geq i}$ eventually decreases strictly. Since $[0, e_n] \setminus [e_n^{1n}, e_n] = [0, v_{n-1} + v_{n-1}^{1n-1}]$ is finite, this sequence eventually lies wholly within $[e_n^{1n}, e_n]$. Using the canonical isomorphism $[e_n^{1n}, e_n] \xrightarrow{\sim} V$ ("lowering subscripts by n ") we find, by the case $n=1$, $j \geq i$ with $e_n \cap a_j \leq e_{n+1} + v_n^{1n}$, hence $a_j \leq e_{n+1} + v_n^{1n} + z_n = e_{n+1} + z_{n+1}$.

c). Again, we use induction on n . If $a \not\leq e_1 + z_1$ then $a \geq f$ as has been noticed at the beginning of the proof of b), hence $e_2 \leq a$. If $n > 1$ we may assume that $a \leq e_1 + z_1$, whence $a = a \cap e_1 + a \cap z_1$. From $a \not\leq e_n + z_n$ we get $a \cap e_1 \not\leq e_n + z_1^{11} + \dots + z_n^{1n-1n-1}$. The induction assumption, applied to $a \cap e_1 \in [e_1^1, e_1] \xrightarrow{\sim} V$ (note that $a \geq e_1^1$ since $V \setminus [e_1^1, e] = [0, v + v^1] \leq e_n$), now implies the assertion.

3.2 An element A of a sublattice $L \subset L(E)$ is called compact if from $A \subset \sum_{i \in I} A_i$ ($A_i \in L$)¹⁾ it always follows that $A \subset \sum_{i \in J} A_i$ for some finite subset $J \subset I$. In the proof of the Congruence Theorem (IV.3) it is crucial that every compact element of $V(E)$ is a (finite) sum of $+$ -irreducible compact elements.

Unfortunately, this is not always true. It may happen, e.g., that $E_i = E_{i+1} + v_i^{11i} \neq E_{i+1}$ for each i , in which case E is not a finite sum of $+$ -irreducible elements. Yet E is compact unless $E = E_\infty + Z$ ($= \bigcap (E_n + Z_n)$ in this case, where Z_n, Z are as defined in (10)). This observation causes us to mainly concentrate on embeddings $\bar{E}$ which satisfy

$$(11) \quad \bigcap_{n \in \mathbb{N}} (E_n + Z) = E_\infty + Z.$$

Such embeddings are called admissible. In the proof of the Congruence

1) We do not require that $\sum_{i \in I} A_i \in L$. Hence our definition differs slightly from that of [5], which is given for abstract lattices.

Theorem (IV.3) there are various other places (besides the just mentioned) where (11) is needed. As to the extent of this restriction, the reader is referred to Section IV.1 below. E.g., if (E, ϕ) is definite then $E_i = E$ for all i (by (3) in Chapt. I) so that (11) is seen to hold trivially. Some remarks on non-admissible embeddings shall be made in a later chapter.

For admissible embeddings we can prove

Theorem 2. Let E be an admissible embedding. Then

- a) If $D \in V(E)$ is compact there is a smallest element $a \in V$ with $v(a) = D$.
- b) Each compact element $D \in V(E)$ is a (finite) sum of +-irreducible compact elements.

Proof a). Assume that the statement is wrong! So there is a strictly decreasing sequence $(a_i)_{i \in \mathbb{N}}$ in V such that $v(a_i) = D$ for all $i \in \mathbb{N}$. By Proposition 1 b) and (11) we have $D \subset \bigcap (E_n + Z) = E_\infty + Z$, and the compactness of D implies that $D \subset E_\infty + Z_n$ for some n . Hence $D = v(a_i) = v(a_i) \cap (E_\infty + Z_n) = v(a_i \cap (e_\infty + z_n))$. But in the interval $[0, e_\infty + z_n]$ we have the minimum condition (Proposition 1 a)), so that a) is valid, contrary to our assumption.

b). If D is +-irreducible there is nothing to be shown. Hence let $D = D_1 + D_2$ with $D_1 \subsetneq D \supsetneq D_2$. For such decompositions (of compact elements D) we prove

$$(12) \quad \begin{aligned} &\text{If } D_1 \text{ is not compact there is } D'_1 \subsetneq D_1 \text{ such that} \\ &D = D'_1 + D_2. \end{aligned}$$

Indeed, let $(A_i)_{i \in I}$ be a family of elements of $V(E)$ with $D_1 \subset \sum_I A_i$ but $D_1 \not\subset \sum_J A_i$ for every finite subset $J \subset I$. By the compactness of D there is a finite subset $J \subset I$ such that $D \subset \sum_J A_i + D_2$, hence

(by distributivity) $D_1 = (\sum_J A_1) \cap D_1 + D_2 \cap D_1$, $D = (\sum_J A_1) \cap D_1 + D_2$.

Choose $D'_1 := \sum_J A_1 \cap D_1$!

Consider now the case where $D \subset \bigcap_{\mathbb{N}} (E_n + Z) = E_\infty + Z$. Again, there is $n \in \mathbb{N}$ such that $D \subset E_\infty + Z_n$, and in $[0, E_\infty + Z_n]$ we have the minimum condition, so that, by (12), there is a decomposition $D = D_1 + D_2$, $D_1 \subsetneq D \not\supseteq D_2$ with D_1, D_2 compact. Now b) follows by an induction argument.

Assume then that $D \not\subset E_n + Z$ for, say, $n \geq n_0$. We contend that for $n \geq n_0$ there are decompositions $D = A_n + D_n$ such that

$$\begin{aligned} (13) \quad & A_n \text{ is a finite sum of } +- \text{irreducible compact elements,} \\ & D_n \subset E_n + Z \\ & A_n \subset A_{n+1} \end{aligned}$$

To prove this we define, for each $r \in \mathbb{N}$ and $\sigma \in \{0,1\}^r$ (a 0,1-sequence of length r), an element $D_\sigma \in V(E)$ such that the following conditions hold. For each $r \in \mathbb{N}$, $D = \sum_{\sigma \in \{0,1\}^r} D_\sigma$ and if D_σ is not compact then $D_\sigma \not\supseteq D_{\sigma 0} = D_{\sigma 1}$; if D_σ is compact but +-reducible then $D_\sigma = D_{\sigma 0} + D_{\sigma 1}$ with $D_{\sigma 0} \subsetneq D_\sigma \not\supseteq D_{\sigma 1}$, and if D_σ is compact and +-irreducible then $D_{\sigma 0} = D_\sigma = D_{\sigma 1}$. Indeed, this is made possible by (12). From Proposition 1 b) (and König's Graph Theorem¹⁾) we learn that there is $r \in \mathbb{N}$ such that each D_σ ($\sigma \in \{0,1\}^r$) turns out either compact and +-irreducible or else is contained in $E_n + Z$. Hence

1) We mean the following (rather trivial) fact: If, in a directed graph where each element has but finitely many successors, there are paths of arbitrary length starting at a given element then there is such a path of infinite length. In the above situation this argument could be avoided by rendering Proposition 1 b) more precise: The 1 there (depending on n) can be chosen so large as to work for every strictly decreasing sequence.

$D = A_n + D_n$ with (13). From $D \not\subseteq E_n + Z$ we get $A_n \not\subseteq E_n + Z$, hence $E_i \subset A_n$ for some i , by Proposition 1 c). For $n \geq n_0 + i$ we find $D \subset A_n + E_n + Z_n = A_n + Z_n$, whence $D = A_n + D \cap Z_n$. But $[0, Z_n]$ is finite (Proposition 1 a)), so that the proof of b) may now be finished in an obvious way.

Remark. If E is not required to be admissible, the assertions of the theorem remain valid for those compact elements $D \in V(E)$ with either $D \subset E_\infty + Z$ or $D \not\subseteq \bigcap_{n \in \mathbb{N}} (E_n + Z)$. For, this is what one really needs in the proofs of a) and b).

As a consequence of part a) of the theorem we get :

Corollary. Each +-irreducible compact element $D \in V(E)$ (where E is admissible) has a greatest predecessor $D_0 \in V(E)$.

Indeed, if d is the smallest element of V with $v(d) = D$ then d is +-irreducible in V and $d \neq v_\infty$ by the compactness of D . From the recursive construction of V it is evident that every such $d \in V$ has a greatest predecessor d_0 in V (we are going to give a list of all such pairs (d, d_0) in a moment); hence $D_0 := v(d_0)$ is the greatest predecessor of D .

In the next chapter we shall need a list of all the possibilities for such (D, D_0) . Since each such pair comes from a pair (d, d_0) in V where $d \in V$ is +-irreducible, $d \neq v_\infty$, and d_0 is the greatest predecessor of d in V , such a list can easily be obtained by recalling the recursive construction of V with the aid of the diagrams of V_2 and V_3 . Note that by 2.3 Corollary 2 we must have $d \in V_{\text{red}}$ or else $d = v_i^{+i+1}$ for some i . Here is the list:

D	R	V	E_{∞}	R_i^{1111}	E_{i+1}^{11}	S_i	V_i^{11}	r_{i+1}^1
D_O	0	R	V_{∞}	R_i	P_i^{1111}	E_{i+1}^{11}	S_i	$V_i + E_{i+1}^{11}$

(120)

D	V_i^{1111}	E_i	F_i	S_i^{11}	E_{i+1}^{1111}	R_i^{11}
D_O	V_{i+1}^{1111}	R_i^{11}	$V_i^{1111} + E_{i+2}$	$F_i + R_{i+1}^{1111}$	$S_i^{11} + E_{i+1}$	E_{i+1}^{1111}

CLASSIFICATION OF ADMISSIBLE EMBEDDINGS

In the course of this chapter the following program is performed: Certain invariants are attached to every admissible embedding E ; it is shown that these invariants characterize E up to congruence and, finally, embeddings with arbitrarily prescribed invariants are constructed. We begin with an introductory discussion of the notion of admissibility.

1. Admissible embeddings

Recall that the embedding E (or the subspace V of (E, Φ)) was called admissible if

$$(1) \quad \bigcap_{n \in \mathbb{N}} (E_n + Z) = E_\infty + Z$$

where $Z = \sum_{j=1}^{\infty} V_j^{1j1j}$. Properly speaking, condition (1) is a requirement of infinite distributivity in the completion $\tilde{V}(E)$ of the lattice $V(E)$ in $L(E)$. Actually (1) will be seen to entail that $\tilde{V}(E)$ is completely distributive¹⁾. It is possible, at this stage, to provide a direct (but somewhat laborious) proof of this fact, but since the transition from $V(E)$ to $\tilde{V}(E)$ would be of very little help in what follows we content ourselves with obtaining the complete distributivity of $\tilde{V}(E)$ as a corollary to the results of this chapter.

The reader may ask for conditions warranting admissibility. As a first example, let us state

1) i.e., for each nonempty family A_{ij} ($i \in I, j \in J$) of elements of $\tilde{V}(E)$ we have $\bigcap_{i \in I} \sum_{j \in J} A_{ij} = \sum_{\varphi: I \rightarrow J} \bigcap_{i \in I} A_{i\varphi i}$ and, of course, the dual equation. See also the beginning of Chapt. IV in [12] where the importance of the completely distributive lattices in the lattice method is explained.

Proposition 1. If $n^-(E, \Phi) < \infty$ then each subspace $V \subseteq E$ is admissible.

Proof. From (3) in I.2 we conclude that $n^-(E_{i+1}, \Phi_{i+1}) \leq n^-(E_i, \Phi_i)$, with equality implying that $E_{i+1} = E_i$. Hence the sequence $E_0 \supset E_1 \supset E_2 \supset \dots$ eventually becomes constant, so that (1) is satisfied in a trivial fashion.

As a consequence of this proposition we shall see in V.2 that, in an arbitrary space (E, Φ) , each maximal positive definite subspace is admissible. It goes without saying that each embedding with finite lattice $V(E)$ is admissible; many naturally distinguished classes of embeddings fall under this specification: e.g., 1-dense embeddings (I.4 and II), 1-closed embeddings (V.1), embeddings of finite codimension (V.4), embeddings with V or $V^\perp$ hyperbolic (V.3). We leave it to the reader to prove that an orthogonal sum of admissible embeddings is again admissible (the converse is not always true). Finally, let us note a lemma which shall be needed in the proof of the Congruence Theorem.

Lemma 1. An embedding $E = (E, \Phi, V)$ is admissible if and only if $E_1 = (E_1, \Phi_1, V_1) / r \Phi_1$ is admissible. Hence, in this case, each $E_i = (E_i, \Phi_i, V_i) / r \Phi_i$ is admissible.

Proof. Since $Z_n \cap E_1 = V_1^{\perp 1 \perp 1} + \dots + V_{n-1}^{\perp n-1 \perp n-1}$ and thus $Z \cap E_1 = \sum_{j \geq 1} V_j^{\perp j \perp j}$ the admissibility of E_1 is equivalent to

$$(1') \quad \bigcap_{n \in \mathbb{N}} (E_n + (Z \cap E_1)) = E_\infty + (Z \cap E_1).$$

Now if E is admissible then (1') is derived from (1) by intersecting both sides with E_1 (and using that $Z = \bigcup Z_i$ and $V(E)$ is distributive). Conversely, if (1') is valid we get (1) by adding $V^{\perp 1}$ on both sides. This time, observe that $V^{\perp 1} + \bigcap_{i \in \mathbb{N}} (E_i + E_1 \cap Z) = \bigcap_{i \in \mathbb{N}} (V^{\perp 1} + E_i + E_1 \cap Z)$ since $V^{\perp 1} \cap (E_i + E_1 \cap Z) = V_1^{\perp 1}$ does not depend on i .

2. Definition of the invariants

2.1 We shall use three kinds of invariants to describe an admissible embedding. First of all we have some obvious cardinal invariants: the indices in the lattice $V(E)$, i.e. the dimensions $\dim B/A$ where $A \subset B$ are elements of $V(E)$. From the recursive construction of $V(E)$ given in Chapter III it is clear - look at the diagrams! - that all these indices (besides $\dim E_\infty/V_\infty$, which shall be taken care of by the third invariant) are determined by the following nine sequences of indices. (We write (B/A) for the dimension of B/A .)

$$(A) \quad \begin{aligned} a_n &= (R_n / r \phi_n) &= (E_n / R_n^{1n}) & \delta_n &= (V_n^{1n1n} / V_{n+1}^{1n+1}) \\ b_n &= (V_n / R_n) & & g_n &= (F_n / V_n^{1n1n} + E_{n+2}) \\ c_n &= (R_n^{1n1n} / R_n) & & h_n &= (S_n^{1n} / F_n + R_{n+1}^{1n+1}) \\ d_n &= (E_{n+1}^{1n} / R_n^{1n1n}) = (R_n^{1n} / E_{n+1}^{1n1n}) & i_n &= (E_{n+1}^{1n1n} / S_n^{1n} + E_{n+1}) \\ e_n &= (r \phi_{n+1} / V_n + E_{n+1}^{1n}) & & & (n \geq 0) \end{aligned}$$

2.2 The second set of invariants consists of

$$(B) \quad \text{the isometry types of the spaces } (V_n, \phi_n) / R_n \quad (n \geq 0).$$

Thus the invariant b_n in (A) is split into the two summands $n^-(V_n, \phi_n)$, $n^+(V_n, \phi_n)$. The reader may wonder why we should stick to the (V_n, ϕ_n) and leave aside the (E_n, ϕ_n) and the (F_n, ψ_n) . As to (E_n, ϕ_n) , iterated application of equation (3) in I.2 yields

$$n^-(E_n, \phi_n) = \sum_{i \geq n} n^-(V_i, \phi_i) + \dim E_n / E_\infty,$$

and a similar equation holds for n^+ , so that the isometry types of the (E_n, ϕ_n) are indeed determined by (A) and (B). As for the (F_n, ψ_n) , observe that $C(F_n, \psi_n, V_n) = C(F_n, \phi_n, V_n) = F_n \cap E_{n+1} = E_{n+2}$ and, on E_{n+2} , $(\psi_n)_{V_n} = \psi_n - \psi_n V_n \psi_n = \psi_n - \phi_n V_n \phi_n = \phi_n - \phi_n V_n^{1n} \phi_n - \phi_n V_n \phi_n = \phi_{n+1} - \phi_n V_n^{1n} \phi_n = \phi_{n+1} - \phi_{n+1} V_n^{1n} \phi_{n+1} = \phi_{n+2}$. Hence from I.2 (3) we conclude that

$n^-(F_n, \psi_n) = n^-(V_n, \psi_n) + n^-(E_{n+2}, \phi_{n+2}) + \dim F_n / E_{n+2}$. Since $(V_n, \psi_n) = (V_n, \phi_n)$ we finally have

$$n^-(F_n, \psi_n) = n^-(V_n, \phi_n) + n^-(E_{n+2}, \phi_{n+2}) + \dim F_n / E_{n+2}.$$

2.3 The third invariant which we must attend to is of a rather different nature (it can only appear if the lattice $V(E)$ is infinite): a hermitean sesquilinear ¹⁾ map $\Omega : E/V_\infty \times E/V_\infty \longrightarrow K^{\mathbb{N}} / K^{(\mathbb{N})}$

($= \prod_{\mathbb{N}} K / \oplus_{\mathbb{N}} K$, $K \supset \mathbb{R}$ the ground field of E). Indeed, the map

$$E_\infty \times E_\infty \longrightarrow K^{\mathbb{N}}, (x, y) \longmapsto (\phi_i(x, y))_{i \in \mathbb{N}}$$

induces such an Ω because V_∞ is the nested union of the $r\phi_i$. Let L be any K -vector space and Q a fixed $\mathbb{R}$ -vector space; notice that $K^{\mathbb{N}} / K^{(\mathbb{N})} = K \otimes_{\mathbb{R}} (\mathbb{R}^{\mathbb{N}} / \mathbb{R}^{(\mathbb{N})})$. There is an obvious notion of isomorphism of hermitean sesquilinear maps

$L \times L \longrightarrow K \otimes_{\mathbb{R}} Q$ which is defined in analogy to the notion of isometry of forms, i.e., to the case $Q = \mathbb{R}$

$$\begin{array}{ccc} L \times L & & \\ \downarrow T \times T & \searrow & \\ L' \times L' & \nearrow & K \otimes_{\mathbb{R}} Q \end{array}$$

(the term "isometry" should be reserved for the case of forms).

If $T: E \xrightarrow{\sim} E'$ is a congruence of embeddings the isomorphism $E_\infty / V_\infty \xrightarrow{\sim} E'_\infty / V'_\infty$ induced by T is an isomorphism from Ω to Ω' . Hence we may choose our third invariant to be

$$(C) \quad \begin{aligned} & \text{the isomorphism type of } \Omega : E_\infty / V_\infty \times E_\infty / V_\infty \longrightarrow K^{\mathbb{N}} / K^{(\mathbb{N})}, \\ & \Omega(x + V_\infty, y + V_\infty) = (\phi_i(x, y))_{i \in \mathbb{N}} + K^{(\mathbb{N})}. \end{aligned}$$

We call Ω the top of the embedding E . We see that the problem of

1) "hermitean sesquilinear" makes sense because $K^{\mathbb{N}} / K^{(\mathbb{N})}$ is a K - K -bimodule and carries the natural involution induced by the involution of K .

classifying admissible embeddings comprises the problem of classifying those $\Omega : L \times L \longrightarrow K^{\mathbb{N}}/K^{(\mathbb{N})}$ occurring as tops of such embeddings. Now, as we shall see in IV.4 below, each Ω defined on a finite dimensional space L will actually occur, up to isomorphism, as the top of a suitable embedding. But already the classification of those Ω with finite dimensional L is to be regarded as a hopeless task (a "wild" problem²⁾), because classifying n -tuples of quadratic forms is a wild problem for $n \geq 3$. Note that, for the sake of classification, we could equally well replace $K^{\mathbb{N}}/K^{(\mathbb{N})} = K \otimes_{\mathbb{R}} (\mathbb{R}^{\mathbb{N}}/\mathbb{R}^{(\mathbb{N})})$ by $K^{\mathbb{N}} = K \otimes_{\mathbb{R}} \mathbb{R}^{\mathbb{N}}$, since $\mathbb{R}^{\mathbb{N}}/\mathbb{R}^{(\mathbb{N})} \cong \mathbb{R}^{\mathbb{N}}$. This means that our problem is equivalent to the problem of classifying sequences of hermitean forms on a vector space of dimension $\leq \aleph_0$, with respect to simultaneous isometry.

For these reasons we have made no attempt to analyze the top of an embedding any further; we shall treat it as a single invariant. The invariants in (A), (B) and (C) are not independent of each other. However, the investigation of the interrelations is deferred to Section IV.4 below, so that we are now ready to state and prove the main result of this chapter.

3. The Congruence Theorem

Theorem 1 (Congruence Theorem for admissible embeddings).

Two admissible embeddings E, E' are congruent if and only if they agree in the invariants (A), (B) and (C).

3.1 For the proof of the nontrivial half of the theorem the reader is assumed to be familiar with the ideas of the lattice method, as

2) We are not willing to enter into philosophical considerations at this point. The reader who is unfamiliar with these ideas may consult the Appendix where we have tried to explain what is meant by a "wild" problem.

sketched in I.3. We shall use the notations introduced there. By $(A) = (A')$ and $(C) = (C')$ there is a canonical lattice isomorphism $V(E) \xrightarrow{\sim} V(E')$ which is given by priming the elements of $V(E)$ (so that there is no need of giving it a name) and which preserves indices and orthocomplementation. Recall that F is the collection of all isomorphisms $T: X \rightarrow X'$ between finite dimensional $X \subset E$, $X' \subset E'$ satisfying the conditions (7), (7') and (9) in I.3.2. Fix an isomorphism $T_\infty: E_\infty/V_\infty \xrightarrow{\sim} E'_\infty/V'_\infty$ of the tops of E and E' . Then we choose, as our candidate for a set of partial isomorphisms, the set G of all $(T: X \rightarrow X') \in F$ which meet the following requirements.

- (1) $(\alpha) \quad T: X \cap E_i \rightarrow X' \cap E'_i$ is isometric for $\phi_i, \phi'_i \quad (i \geq 0)$,
 $(\beta) \quad T: X \cap F_i \rightarrow X' \cap F'_i$ is isometric for $\psi_i, \psi'_i \quad (i \geq 0)$,
 (2) $(\gamma) \quad T: X \cap E_\infty / X \cap V_\infty \rightarrow X' \cap E'_\infty / X' \cap V'_\infty$ coincides with the restriction of T_∞ ,
 (3) if $\dim V_i/R_i < \infty$ then $V_{i-1}^{i-1} \subset S_{i-1} + X \quad (V \subset R+X \text{ if } i=0)$.

Of course, since $T \in F$, (3) implies the corresponding condition in E' . Note that from $V_{i-1}^{i-1} \subset S_{i-1} + X$ it follows that also $V_i \subset R_i + X$. Furthermore, we remark that if $b_i = \dim V_i/R_i < \infty$ for some i then $b_j = 0$ for $j \geq i+2$. Indeed, $V_{i+1} = r\phi_{i+1} + V_i^{i-1} = R_i^{i-1} = E_{i+1}$ in this case, whence $E_{i+2} = r\phi_{i+2}$. We see that condition (3) involves at most two numbers i and is empty if $V(E)$ is infinite. In this case we have $(0 \rightarrow 0) \in G$.

We have to show that G is nonempty and has the Ping-Pong property (PP) in I.3.1. For technical reasons it is expedient to defer the proof of nonemptiness until later (3.11); so let us tackle the verification of (PP) first.

3.2 Thus, let $(T: X \rightarrow X') \in G$ and $y \in E \setminus X$ be given. We contend that the filter $M = \{A \in V(E) \mid y \in X + A\}$ in $V(E)$ has a smallest element. In fact, we have seen in I.3.2 that $y \in X + D$ with $D = D(X, y) := \bigcap M$, whence it is enough to show that $D \in V(E)$. This is clear if the minimum condition holds in M . Assume then that there are infinite strictly descending sequences in M . By III.3 Proposition 1 b) this means that $D \subset \bigcap_{n \in \mathbb{N}} (E_n + Z)$, hence $D \subset E_\infty + Z$ since E is admissible. Since $Z = \bigcup Z_n$ there is $n \in \mathbb{N}$ such that $y \in X + E_\infty + Z_n$, i.e. $E_\infty + Z_n \in M$. But in $[0, E_\infty + Z_n]$ the minimum condition holds (III.3 Prop. 1 a)), so that D is the smallest element of $M \cap [0, E_\infty + Z_n]$. This proves our contention. For the proof of (PP) we may assume w.l.o.g. that $y \in D = D(X, y)$. We note also that D is a compact element of $V(E)$ in the sense of III.3.2.

Next we claim that it suffices to prove the following property

(4) If D is $\pm$ -irreducible there is $d \in D_0 \cap V_\infty$ and an extension $T_1: X \oplus K(y+d) \rightarrow X'_1$ of T belonging to G .

Here D_0 is the greatest predecessor of D , which exists by the corollary to Theorem 2 in III.3. Let us show how (PP) follows from (4).

1st case: The minimum condition holds in $[0, D]$. Assume that (PP) is violated for T, y . Among all situations $T_0: X_0 \rightarrow X'_0, y_0$ with $T_0 \in G, D(X_0, y_0) \subset D$ and where (PP) is violated, choose one with $D(X_0, y_0)$ minimal. Then $D(X_0, y_0)$ must be $\pm$ -irreducible (otherwise we could prove (PP) for T_0, y_0 in two steps). Now it is easy to get a contradiction, using (4) and the minimality of $D(X_0, y_0)$.

2nd case: D is $\pm$ -irreducible. Consider the extension T_1 of T onto $X_1 := X \oplus K(y+d)$ warranted by (4), where $d \in D_0 \cap V_\infty$. By the first case we may extend T_1 to $(T_2: X_2 \rightarrow X'_2) \in G$ with $d \in X_2$, since in

$[0, V_\infty]$ we have the minimum condition. Hence (PP) is proved in the present case. For the third case we shall need the more precise fact that the restriction of T_2 to $X \oplus Ky$ also belongs to G . This follows from the lemmas in I.3.2, if we observe that $T_2 y \in D' \setminus (X' + D'_0)$ (since $T_2 \in G$) so that $D(X', T_2 y) = D'$, and that the filter $M = [D, E]$ is prime (since D is $+-$ irreducible and $V(E)$ is distributive).

3rd case: D is arbitrary. By III Theorem 2 we may write D as a sum of $+-$ irreducible elements, $D = D_1 + \dots + D_r$, and we may assume that $D_i \not\subseteq \sum_{j \neq i} D_j$ for $i=1, \dots, r$. Decompose $y = y_1 + \dots + y_r$ with $y_i \in D_i$. Suppose that we have already extended T to $T_{s-1} \in G$ on $X_{s-1} \supset X$ such that $y_1, \dots, y_{s-1} \in X_{s-1} \subset X + D_1 + \dots + D_{s-1}$, for some $s \geq 1$. Then $\bar{D}_s := D(X_{s-1}, y_s) \subset D_s$ and $D = D_1 + \dots + \bar{D}_s + \dots + D_r$. Hence $D_s = D_1 \cap D_s + \dots + \bar{D}_s + \dots + D_r \cap D_s$, and therefore $D_s = \bar{D}_s$ since $D_i \cap D_s \neq D_s$ for $i \neq s$. By the second case above we may extend T_{s-1} onto $X_s := X_{s-1} \oplus Ky_s$, and the inductive assumption $y_1, \dots, y_s \in X_s \subset X + D_1 + \dots + D_s$ is again valid. Thus we end up with $y \in X_r$.

Thereby the proof of (PP) is reduced to the proof of (4).

3.3 In view of the definition of G and of I.3.2 we may reformulate (4) in the following way. We are given $(T: X \rightarrow X') \in G$, a compact $+-$ irreducible element $D \in V(E)$ and a vector $y' \in D' \setminus (X' + D'_0)$. We are to search for $d' \in D'_0 \cap V_\infty$ and $y \in D \setminus (X + D_0)$ such that ¹⁾

$$(5) \quad \text{if } D \subseteq E_1 \text{ then } {}^2) \quad \begin{aligned} \phi_1(y, x) &= \phi_1'(y' + d', Tx) \quad \text{for all } x \in X \cap E_1, \\ \phi_1(y, y) &= \phi_1'(y' + d', y' + d') \end{aligned}$$

1) We have interchanged the roles of y and y' in order to prevent an excessive increase of primes.

2) Note that $D \subseteq E_1$ is equivalent to $y \in E_1$ (or $y' \in E'_1$). One direction is clear since $y \in D$. Conversely, if $y \in E_1$ then $D \subseteq E_1$ since $D = D(X, y)$.

- (6) if $D \subset F_i$ then $\Psi_i(y, x) = \Psi'_i(y' + d', Tx)$ for all $x \in X \cap F_i$
 $\Psi_i(y, y) = \Psi'_i(y' + d', y' + d')$,
- (7) if $D \subset E_\infty$ then $T_\infty(y + V_\infty) = y' + V'_\infty$.

In fact, the extension $T_1: X_1 = X \oplus KY \longrightarrow X'_1 = X' \oplus K(y' + d')$, $y \mapsto y' + d'$ of T belongs to F by I.3.2 (note that $D(X', y' + d') = D(X', y') = D'$, $D(X, y) = D$), and (5), (6), (7) make sure that T_1 satisfies (2) in lieu of T (note that for $A \in V(E)$ we have $A \cap X_1 = A \cap X$ if $D \not\subset A$ and $A \cap X_1 = (A \cap X) \oplus KY$ if $D \subset A$). Hence T_1 belongs to G .

Now, the existence of y and d' has to be checked separately in each of the fourteen possibilities for (D, D_0) listed at the end of Chapter III. The general procedure will be to choose a preliminary $y \in D \setminus (X + D_0)$ and then to apply various corrections to both y and y' (adding vectors from $D_0 \cap V_\infty$ and $D'_0 \cap V'_\infty$ to y and y'). By abuse of notation we shall not introduce new symbols for the corrected vectors (hence the d' above will disappear from the formulas). The quantities appearing in (5) shall be referred to as the " ϕ_i -quantities", more exactly, the " ϕ_i -angles" of y on $X \cap E_i$ and the " ϕ_i -length" of y ; similarly with (6). E.g., to adjust the ϕ_i -angles of y modulo B means to replace y by a vector $y + b$ ($b \in B$) with $\phi_i(y + b, x) = \phi'_i(y', Tx)$ for $x \in X \cap E_i$. In following our arguments the reader will have to make extensive use of the diagrams of $V_1(E)$, $V_2(E)$, $V_3(E)$ and $V_{\text{red}}(E)$ given in Chapter III. Such facts as $r\phi_i \subset r\phi_j$ for $i \leq j$ and $r\Psi_i \subset r\Psi_j$ for $i+2 \leq j$ will be used freely.

Let us now start the discussion of the individual cases.

3.4 $D = R$. This case is covered by I.3 Lemma 3, because R is totally isotropic and lies in the radical of each ϕ_i ($i \geq 1$) and Ψ_i ($i \geq 0$).

3.5 $D = V$ ($D_0 = R$). Here, too, we need only care about the ϕ_0 -quantities, because the ψ_0 -quantities (where defined) coincide with the ϕ_0 -quantities by I.2 (1). We shall reduce this case to the extension lemma II.1. If $R=0$ we proceed as follows. Decompose $X = (X \cap V^\perp) \oplus U$ with $U \supset X \cap V$ and $X' = (X' \cap V'^\perp) \oplus TU$, and consider the embeddings $V \subset V+U$, $V' \subset V'+TU$. They are 1-dense: $V^\perp \cap (V+U) \subset (V^\perp + X) \cap (V+X) = R+X = X$, hence $V^\perp \cap (V+U) = V^\perp \cap (V+U) \cap X = V^\perp \cap ((V \cap X) + U) = V^\perp \cap U = 0$. Furthermore conditions a) b) of the extension lemma are easily seen to hold for $T:U \rightarrow TU$. If $R \neq 0$, we first consider the embeddings $V/R \subset R^\perp/R^{\perp\perp}$, $V'/R' \subset R'^\perp/R'^{\perp\perp}$ and the induced map $X \cap R^\perp/X \cap R^{\perp\perp} \rightarrow X' \cap R'^\perp/X' \cap R'^{\perp\perp}$. By the case $R=0$ just done there is $y \in V \setminus (R+X)$ with the correct ϕ_0 -angles on $X \cap R^\perp$ and the correct ϕ_0 -length. The remaining ϕ_0 -angles can be adjusted modulo R : Put $X = X \cap R^\perp \oplus U$ and observe that $E = R+U^\perp$. (Similar arguments shall frequently be used; corrections modulo $A \subset E$ are sufficient to adjust the ϕ_0 -angles on a supplement of $A^\perp \cap X$ in X .)

3.6 $D = E_\infty$ ($D_0 = V_\infty$). We choose a preliminary y with property (7); then $y \in E_\infty \setminus (X+V_\infty)$. Since T_∞ is an isomorphism of tops we have $\phi_i(y, x) = \phi_i'(y', Tx)$ for $x \in X \cap E_\infty$ and $\phi_i(y, y) = \phi_i'(y', y')$, for large i . But $X \cap E_i = X \cap E_\infty$ for large i , so that (5) is seen to hold for large i . Furthermore, by III Lemma 1 e) we have $\psi_i = \phi_i - \phi_{i+1} + \phi_{i+2}$ on E_∞ , and $X \cap F_i = X \cap E_\infty$ for large i , whence (6) holds as well, for large i . Now we are in a position to apply the following lemma, which will be used frequently.

Lemma 2. Assume that $D \subset E_{n+2}$ and $D_0 \supset V_n + V_n^{\perp n}$. If $y \in D \setminus (X+D_0)$ is chosen such as to have the correct ϕ_i - and ψ_i -quantities for $i \geq n+1$ (if defined, i.e., if $D \subset E_i$ ($D \subset F_i$))¹⁾, with the possible

1) See footnote 2) in 3.3

exception of the ϕ_{n+1} - and ψ_{n+1} -length, then it is possible to adjust y and y' modulo $D_0 \cap V_\infty$ and $D'_0 \cap V'_\infty$ such that (5) and (6) hold for all i .

Proof. By induction the problem is immediately reduced to the case $n=0$. If $n=0$, we begin by adjusting the ψ_0 -angles of y . They are already correct on $X \cap r\psi_0 = X \cap V^\perp = X \cap F_0 \cap V^\perp \psi_0$, so that they may be adjusted modulo $V \subset r\phi_1 \cap r\psi_1 \cap r\psi_2 \cap D_0$. Next we try to arrange the ϕ_0 -angles. If we do not want to destroy what we have already achieved, we may apply further corrections only modulo

$$W := V \cap (X \cap F_0)^\perp \psi_0 + V^\perp \cap (X \cap E_1)^\perp \cap (X \cap F_1)^\perp \psi_1 = V \cap (X \cap F_0)^\perp + V^\perp \cap (X \cap E_1)^\perp.$$

Such corrections suffice to adjust the ϕ_0 -angles if those on $X \cap W^\perp$ are already correct. But $X \cap W^\perp = X \cap (V^\perp + (X \cap F_0)) \cap (V^\perp + (X \cap E_1)) =$
 $= X \cap F_0 \cap (V^\perp + (X \cap E_1)) = X \cap E_2 + X \cap V^\perp$. On E_2 we have $\phi_0 = \psi_0 + \phi_1 - \phi_2$, whence the ϕ_0 -angles of y on $X \cap E_2$ are correct. Since the ϕ_0 -angles of y on $X \cap V^\perp$ coincide with the ψ_0 -angles they, too, need no correction. Finally we have to care about the lengths of y for $\phi_1, \phi_0, \psi_1, \psi_0$. First we adjust the ϕ_1 -length either of y or of y' by adding a suitable vector from $V^\perp \cap Y^\perp$ (orthogonal to X for $\phi_0, \phi_1, \psi_0, \psi_1$, lest we should affect the angles). Such a correction is possible as long as $\dim V^\perp/S = \dim V_1/R_1$ is infinite; e.g., if $n^-(V_1, \phi_1) = \infty$ and $\phi_1(y, y) < \phi_1(y', y')$ the correction may be applied to y' . If $\dim V_1/R_1 < \infty$ there is, by (3), a ϕ_1 -orthogonal basis of a supplement of R_1 in V_1 consisting of vectors $x_1, \dots, x_m \in X$. Then $\phi_1(y, y) = \phi_2(y, y) +$
 $+ \sum_{i=1}^m \phi_1(y, x_i) \phi_1(x_i, x_i)^{-1} \phi_1(x_i, y) = \phi_2(y', y') + \sum_{i=1}^m \phi_1(y', Tx_i) \phi_1(Tx_i, Tx_i)^{-1} \phi_1(Tx_i, y') =$
 $= \phi_1(y', y')$; no correction is needed. If the ψ_1 -length of y is defined, i.e., if $D \subset F_1 \cap E_2 = E_3$, then $\psi_1(y, y) = (\phi_1 - \phi_2 + \phi_3)(y, y) =$
 $= \psi_1(y', y')$. The ϕ_0 - and ψ_0 -lengths of y can be arranged in the same fashion modulo V , which does not affect the ϕ_1 - and ψ_1 -lengths.

3.7 $D = R_n^{1n1n}$ ($D_0 = R_n$). By considering $E_n/r\phi_n$, $X \cap E_n/X \cap r\phi_n$, etc. and applying the same arguments as in 3.4 we find $y \in D \setminus (X + D_0)$ with the correct ϕ_i - and ψ_i -quantities for $i \geq n$. In order to be able to apply Lemma 2 we must adjust the ϕ_{n-1} - and ψ_{n-1} -angles (we have $D_0 = R_n \supset V_{n-2} + V_{n-2}^{1n-2}$). Since $R_n^{1n1n} \subset (V_{n-1}^{1n-1})^{1n-1}$ the ψ_{n-1} -angles of y on $X \cap F_{n-1} \subset X \cap E_{n-1}$ coincide with the ϕ_{n-1} -angles; hence it suffices to adjust these. The ϕ_{n-1} -angles of y on $X \cap V_{n-1}^{1n-1}$ are (and have to be) zero because $R_n^{1n1n} \subset V_{n-1}^{1n-1}$, whence the remaining ϕ_{n-1} -angles may be adjusted modulo $V_{n-1} \subset r\phi_n \cap r\psi_n \cap r\psi_{n+1} \cap D_0$. Now, if $n \geq 2$ apply Lemma 2 and if $n = 1$ adjust the ϕ_0 -length (= ψ_0 -length) in the same manner as at the end of 3.6.

3.8 The cases $D = E_{n+1}^{1n}$ and $D = S_n$ are treated exactly in the same fashion as the preceding case. However, when $D = S_n$ we have no longer $D \subset r\phi_{n+1}$; but the ϕ_{n+1} -quantities coincide with the ϕ_n -quantities, since $S_n \supset V_n$.

3.9 $D = V_n^{1n}$ ($D_0 = S_n$). Consider first the case $n = 0$. The arguments of 3.5 applied to V^1 in lieu of V (here we need that T respects ψ_0) yield a vector $y \in V^1 \setminus (X + S)$ with the correct ϕ_0 -quantities. For $i \geq 2$ we have $V^1 \subset r\phi_i \cap r\psi_i$, and the ϕ_i - and ψ_i -quantities coincide with the ϕ_0 -quantities. If n is arbitrary we find $y \in V_n^{1n} \setminus (X + S_n)$ with the correct ϕ_i - and ψ_i -quantities for $i \geq n$. Now we may continue in the same way as in 3.7.

3.10 $D = r\phi_{n+1}$ ($D_0 = E_{n+1}^{1n} + V_n$). Again, consider $E_n/r\phi_n$. By I.3 Lemma 3 we find $y \in D \setminus (X + D_0)$ with the correct ϕ_n -angles. We have $D = r\phi_{n+1} \subset r\psi_{n+1} \cap r\psi_{n+2}$ and $D_0 \supset V_{n-1} + V_{n-1}^{1n-1}$; hence before applying Lemma 2 (in case $n \geq 1$) we must only adjust the ψ_n -angles, if necessary. But they coincide with the ϕ_n -angles on $X \cap F_n \subset X \cap E_n$, since $r\phi_{n+1} \supset V_n^{1n}$. In case $n = 0$ the ϕ_0 -length (= ψ_0 -length) is

again adjusted modulo $V \subset D_0$.

3.11 Let us pause for a moment to insert the still missing proof that $G \neq \emptyset$. Let m be the smallest natural number with $\dim V_m/R_m = b_m$ finite and $\neq 0$ (if there is no such number we have $(0 \rightarrow 0) \in G$). As remarked in 3.1 we then have $b_i = 0$ for $i \geq m+2$. We shall construct $(T: X \rightarrow X') \in G$ such that $X = X_1 \oplus X_2$, $V_{m-1}^{i_{m-1}} = S_{m-1} \oplus X_1$ ($V = R \oplus X_1$ if $m=0$), and $V_m^{i_m} = S_m \oplus X_2$ if $b_{m+1} < \infty$, $X_2 = 0$ if $b_{m+1} = \infty$. Now observe that in the cases treated so far, 3.4-3.10, condition (3) was used only if $D_0 \supset V_m$. But as long as $D \subset V_{m-1}^{i_{m-1}}$ ($D \subset V$ if $m=0$) or $D \subset V_m^{i_m}$ we have $D_0 \subset S_m$, which excludes the possibility that $D_0 \supset V_m$, since $V_m \not\subset S_m$. Hence choose a supplement Y_1 of S_{m-1} in $V_{m-1}^{i_{m-1}}$ (of R in V if $m=0$) and a supplement Y_2 of S_m in $V_m^{i_m}$, if $b_{m+1} < \infty$. Start with $X=0$ and successively "adjoin" to X vectors y out of a basis of Y_1 and Y_2 , modulo S_{m-1} (R if $m=0$) and S_m . This is made possible by the above observation, and since $V_{m-1}^{i_{m-1}}$ (V if $m=0$) and $V_m^{i_m}$ (if $b_{m+1} \neq 0$) are +-irreducible, and $D = V_{m-1}^{i_{m-1}}$ (V if $m=0$) if $y \in Y_1$ and $D = V_m^{i_m}$ if $y \in Y_2$.

3.12 In the verification of (4), we now come to those cases where $D \not\subset E_\infty$. We begin with the case $D = V_n^{i_n i_n}$ ($D_0 = V_{n+1}^{i_{n+1}}$). By I.3 Lemma 3 there is $y \in D \setminus (X + D_0)$ with the correct ϕ_n -angles. For $i \geq n+1$, the ϕ_i - and ψ_i -quantities are not defined, because $D \not\subset E_i \supset F_i$. The ψ_n -quantities coincide with the ϕ_n -quantities, since $V_n^{i_n i_n} \supset V_n^{i_n}$. Although $D_0 \supset V_{n-1} + V_{n-1}^{i_{n-1}}$ we cannot apply Lemma 2, since $D \not\subset E_{n+1}$. Thus we go on by adjusting first the ϕ_n -length (= ψ_n -length) modulo $V_n \subset D_0$ (we have $\dim V_n/R_n = \infty$, because $R_n^{i_n} \supset D \not\subset E_{n+1}$); then, if $n \geq 1$, the ϕ_{n-1} -angles modulo $V_{n-1} \subset R \cap D_0$ (those on $X \cap V_{n-1}^{i_{n-1}}$ are already correct, namely, equal to the corresponding ϕ_n -angles). The ψ_{n-1} -quantities are not defined,

since $F_{n-1} \cap D \subset D_0$, hence $D \not\subset F_{n-1}$. If $n=1$ there remains to adjust the ϕ_0 -length modulo V , and if $n>1$ Lemma 2 applies.

3.13 $D = F_n$ ($D_0 \supset E_{n+2}$). By applying I.3 Lemma 3 to $F_n/r\Psi_n$, $X \cap F_n/X \cap r\Psi_n$ etc. we find $y \in D \setminus (X + D_0)$ with the correct Ψ_n -angles. The Ψ_n -length may be adjusted modulo V_n (again, $\dim V_n/R_n = \infty$; otherwise, $V_n + V_n^{in} = E_{n+1} = E_{n+2} = F_n$, a contradiction to $D_0 \supset E_{n+2}$), and the ϕ_n -angles can be arranged modulo $V_n^{in} = r\Psi_n$ (they are already correct on $X \cap V_n^{in}$, namely, equal to the Ψ_n -angles), whereas for $i \geq n+1$ the ϕ_i - and Ψ_i -quantities are not defined. This time, the ϕ_n -length has to be corrected modulo $r\Psi_n = V_n^{in}$, which is possible if $\dim V_n^{in}/S_n = b_{n+1}$ is infinite. But if $b_{n+1} < \infty$ the ϕ_n -length needs no correction, because $\phi_n(y, y) = \Psi_n(y, y) + \sum \phi_n(y, x_i) \phi_n(x_i, x_i)^{-1} \phi_n(x_i, y)$ for some ϕ_n -orthogonal basis (x_i) of a supplement of S_n in V_n^{in} with $x_i \in X$, by (3). Now we are in the same situation as in 3.12, since the Ψ_{n-1} -quantities are, again, not defined ($D \subset F_{n-1}$ would imply that $D \subset F_{n-1} \cap F_n = E_{n+2}$).

3.14 $D \in \{S_n^{in}, E_{n+1}^{in}, R_n^{in}\}$ ($D_0 \supset F_n + R_{n+1}^{in}$). These cases are handled quite similarly to 3.12, apart from the fact that we do not have to care about the Ψ_n -quantities: they are not defined, since $D_0 \supset F_n$. Indeed, the relations $D \not\subset E_i \supset F_i$ for $i \geq n+1$, $R_n^{in} \supset D \not\subset E_{n+1}$ and $D \not\subset F_{n-1}$ used in 3.12 are easily verified in the present situation.

3.15 $D = E_n$ ($D_0 = R_n^{in}$). The difference to 3.14 consists in the fact that R_n^{in} may happen to equal E_{n+1} , so that $b_n < \infty$ is no longer excluded. This means that we are not able to adjust the ϕ_n -length of y modulo V_n (by adding to y vectors which are ϕ_n -perpendicular to y). But we may proceed as follows. Since $y \notin R_n^{in} + X \cap E_n = (R_n \cap (X \cap E_n)^{in})^{in}$ there is $z \in R_n \cap (X \cap E_n)^{in}$ with $\phi_n(y, z) = 1$.

Hence each $y + \lambda z$ has the correct ϕ_n -angles, and

$$\phi_n(y + \lambda z, y + \lambda z) = \phi_n(y, y) + 2\lambda. \text{ Choose } \lambda = \frac{1}{2}(\phi'_n(y', y') - \phi_n(y, y)).$$

The proof of the Congruence Theorem is thus complete. As a first corollary we have the following result which, for embeddings with finite lattice, is equivalent to the Congruence Theorem.

Corollary. For admissible embeddings, $E \cong E'$ if and only if

- a) $(a_0, b_0, \dots, i_0) = (a'_0, b'_0, \dots, i'_0)$
- b) $(V, \phi) \cong (V', \phi')$
- c) $E_1 \cong E'_1$ (where $E_1 = (E_1, \phi_1, V_1) / r \phi_1$).

Remark. The question arises if, under the proviso of a) b) c), each congruence $\bar{T}: E_1 \xrightarrow{\sim} E'_1$ can be lifted to a congruence $E \cong E'$. The answer is no in general, but yes if $E_1^{11} = V^{11} + E_1$ (i.e. $g_0 = h_0 = i_0 = 0$; compare the diagram of $V_1(E)$, Figure 5 below), even for non-admissible embeddings. To explain this, we mention the following

Criterion: $\bar{T}$ may be lifted to a congruence $E \cong E'$ if and only if the isomorphism $T_0: V^1/E_1^1 \rightarrow V'^1/E_1'^1$ induced by $\bar{T}$ transforms each of the topologies ¹⁾ $\sigma(V^1/E_1^1, \phi, E_1^{11})$, $\sigma(V^1/E_1^1, \phi, S^1)$, $\sigma(V^1/E_1^1, \phi, F)$ on V^1/E_1^1 into the corresponding topology on $V'^1/E_1'^1$.

Now, if $E_1^{11} \subset V^{11} + E_1$ this condition is automatically satisfied, since $\phi_1|_{E_1 \times V^1} = \phi|_{E_1 \times V^1}$ and $\bar{T}$ is a congruence. On the other hand, if $E_1^{11} \not\subset V^{11} + E_1$ it is always possible, using the methods of the next section, to construct $\bar{T}$ which violate the condition. As to the proof of this criterion, a few hints must suffice. The condition on the topologies means that T_0 admits a "transpose" isomorphism $T_0^t: E_1'^{11}/V'^{11} \xrightarrow{\sim} E_1^{11}/V^{11}$ (i.e. for $x' \in E_1'^{11}$, $y \in V^1$ we have

¹⁾ $\sigma(A, \phi, B)$ denotes the weak linear topology on A induced by the (possibly degenerate) pairing $\phi: A \times B \rightarrow K$; see [12], Chap. X.

$\Phi(T_O^t(x' + V^{11}), y + V^1) = \Phi(x' + V^{11}, T_O(y + V^1))$ which maps S^1/V^{11} and F/V^{11} onto S^1/V^{11} and F/V^{11} . On $E_1 + V^{11}/V^{11} = E_1/V_1^{11}$, $(T_O^t)^{-1}$ coincides with the map induced by $\bar{T}$, hence $(T_O^t)^{-1}$ respects all the spaces A/V^{11} with A in the interval $[V^{11}, E_1^{11}]$ of $V_1(E)$. As a candidate for a set of partial isomorphisms we choose the set G of all $(T: X \rightarrow X') \in F$ which are Φ -isometries and which, on $X \cap E_1/X \cap \Phi_1$, $X \cap E_1^{11}/X \cap V^{11}$, coincide with the restrictions of $\bar{T}$, $(T_O^t)^{-1}$ respectively.

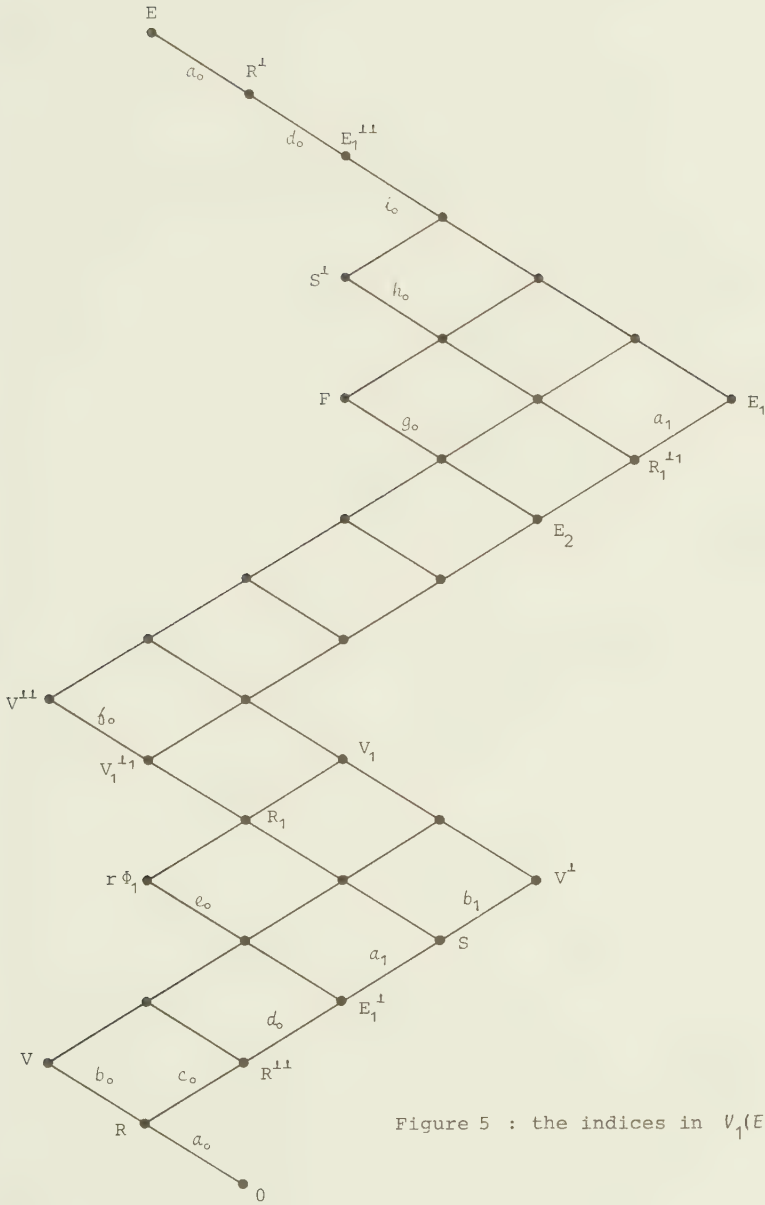
4. Construction of embeddings with prescribed invariants

4.1 Interdependences among the invariants. We are going to establish certain conditions to which the invariants are subject. Hereby, as well as in subsequent constructions, it will be useful to have the diagram in Figure 5 at hand.

Lemma 3. Let E be an arbitrary embedding and $n, j \in \mathbb{N}$.

- i) $a_n < \infty \Rightarrow c_n = 0$ and (if $n \geq 1$) $i_{n-1} = 0$
- ii) $b_n < \infty \Rightarrow$
- | | |
|-----------------------------|------------------|
| $g_j = h_j = 0$ | for $j \geq n-1$ |
| $d_j = e_j = f_j = i_j = 0$ | for $j \geq n$ |
| $a_j = c_j = 0$ | for $j \geq n+1$ |
| $b_j = 0$ | for $j \geq n+2$ |
- iii) $(V_n, \Phi_n)/R_n$ hyperbolic $\Rightarrow$
- | | |
|-----------------------------------|-------------------------------|
| $g_j = 0$ | for $j = n-1$ or $j \geq n+1$ |
| $e_j = i_j = 0$ | for $j \geq n$ |
| $a_j = c_j = d_j = f_j = h_j = 0$ | for $j \geq n+1$ |
| $b_j = 0$ | for $j \geq n+2$ |

Proof. It suffices to consider the cases $n=0$ and $n=1$. The first statement in i) is clear; if $a_1 = \dim S^1 + E_1/S^1 < \infty$ then $S^1 + E_1$ is μ -closed, whence $i_0 = 0$. The assertions in ii) follow from

Figure 5 : the indices in $V_1(E)$

$b_0 < \infty \Rightarrow R^1 = V + V^1$ and $b_1 < \infty \Rightarrow S^1 = V^1 + V^{11}$. Finally, iii) is deduced from $(V, \Phi)/R$ hyperbolic $\Rightarrow E_1 = V + V^1 \Rightarrow E_1^{11} = S^1$ and from $(V_1, \Phi_1)/R_1 = (V^1, \Phi)/S$ hyperbolic $\Rightarrow F = V^1 + V^{11}$.

We shall now describe a connection between the top of E and the $(V_n, \Phi_n)/R_n$. Consider first an arbitrary hermitean sesquilinear map $\Omega: \bar{L} \times \bar{L} \rightarrow K^{\mathbb{N}}/K^{(\mathbb{N})}$. A sequence $(\Omega_i)_{i \in \mathbb{N}}$ of hermitean forms on L is called a lifting of Ω , if there is $\bar{T}: L \xrightarrow{\sim} \bar{L}$ which is an isomorphism between $p \circ (\Omega_i)_{i \in \mathbb{N}}$ and Ω , where $p: K^{\mathbb{N}} \rightarrow K^{\mathbb{N}}/K^{(\mathbb{N})}$ is the canonical projection.

$$\begin{array}{ccc} L \times L & \xrightarrow{(\Omega_i)_{i \in \mathbb{N}}} & K^{\mathbb{N}} \\ \downarrow \bar{T} \times \bar{T} & & \downarrow p \\ \bar{L} \times \bar{L} & \xrightarrow{\Omega} & K^{\mathbb{N}}/K^{(\mathbb{N})} \end{array}$$

Each Ω admits a lifting.

Lemma 4. If Ω is the top of an embedding E then Ω admits a lifting $(\Omega_i)_{i \in \mathbb{N}}: L \times L \rightarrow K^{\mathbb{N}}$ such that

$$(8) \quad \text{for each } i \in \mathbb{N}, (L, \Omega_i - \Omega_{i+1}) \text{ is isometric to a subspace of } (V_i, \Phi_i)/R_i.$$

Proof. Choose L to be a supplement of V_∞ in E_∞ and let $\bar{T}: L \xrightarrow{\sim} E_\infty/V_\infty$ be the canonical isomorphism. Put $\Omega_i := \Phi_i|_{L \times L}$. By the definition of the top, $(\Omega_i)_{i \in \mathbb{N}}$ is a lifting of Ω . For the proof of (8) we may assume that $n^-(V_i, \Phi_i) < \aleph_0 = n^+(V_i, \Phi_i)$ (for the particular i under consideration), because if $(V_i, \Phi_i)/R_i$ is hyperbolic or finite dimensional then $L = 0$ by Lemma 3. Thus $\Omega_i - \Omega_{i+1} = \Phi_i|_{V_i} - \Phi_{i+1}|_{V_i} = \chi_+ + \chi_-$ with χ_+ positive definite and χ_- negative of rank $\leq n^-(V_i, \Phi_i)$ (cf. the beginning of II.3.5 where we have introduced the χ -forms). On each totally isotropic subspace M of $(L, \Omega_i - \Omega_{i+1})$ we have that

$\chi_- = -\chi_+$ is definite, whence $\dim M \leq n^-(V_1, \phi_1)$. From this (8) follows at once.

As an example, consider an embedding E with (E, ϕ) positive definite. Then all $(E_i, \phi_i)/r\phi_i$ are positive definite as well, hence (8) means $\Omega_i(x, x) > \Omega_{i+1}(x, x) > 0$ for all $x \in L$ and $i \in \mathbb{N}$.

In the next two subsections, our aim is to show by construction that the restrictions imposed upon the invariants by the above two lemmas are in fact the only relations between them.

4.2 Embeddings without top. In the constructions we are going to perform, $\perp$ -dense embeddings will serve as the building blocks, in a certain sense. Therefore it seems worth-while to first have a closer look at these. Let (V, ϕ) be a positive definite space of dimension $\aleph_0$, and let $V^\times$ be its algebraic dual, turned into a K -left vector space by $\lambda f := f\lambda^*$ ($\lambda \in K, f \in V^\times$). Let $\tilde{V} \subset V^\times$ be the subspace consisting of the functionals which are continuous with respect to the norm topology. $V^\times$ is made into a Hilbert space by $\tilde{\phi}(f, g) := \sum_{i \in \mathbb{N}} f(v_i)^* g(v_i)$ ($f, g \in \tilde{V}$), where $\{v_i\}_{i \in \mathbb{N}}$ is an arbitrary orthonormal basis of V ($\tilde{\phi}$ is independent of its choice), and the map $\mu: V \rightarrow \tilde{V}, v \mapsto \phi(-, v)$ is an isometric embedding. If we now choose $Q \subset V^\times$ with $\dim Q \leq \aleph_0$ and $Q \cap \mu V = 0$, and put

$$E := V \oplus Q, \quad \phi(v, q) := q(v) \quad (v \in V, q \in Q), \quad \phi|_{Q \times Q} \text{ arbitrary}$$

then (E, ϕ, V) is a $\perp$ -dense embedding. Furthermore each $\perp$ -dense embedding E with V positive definite can be obtained in this way:

$E \hookrightarrow V^\times, x \mapsto \phi(-, x)$, and $\phi_1(x, y) = \phi(x, y) - \tilde{\phi}(\phi(-, x), \phi(-, y))$ for $x, y \in E_1$. Let us call such an embedding parseval, if $E_1 = E = r\phi_1$ (i.e. $Q \subset \tilde{V}$ and $\phi|_{Q \times Q} = \tilde{\phi}|_{Q \times Q}$; in all of E we have Parseval's identity $\phi(x, y) = \sum \phi(x, v_i) \phi(v_i, y)$), of finite type if $E_1 = E$ (i.e. $Q \subset \tilde{V}$)

and of infinite type if $E_1 = V$ (i.e. $Q \cap \tilde{V} = 0$). Because $\dim V^x / \tilde{V} = 2^{\aleph_0} = \dim \tilde{V} / \mu V$ we shall have a considerable amount of elbow-room for our constructions.

Theorem 2. Let $\hat{E} = (\hat{E}, \hat{\Phi}, \hat{V})$ be an arbitrary embedding. Then there is an embedding E such that $E_1 \cong \hat{E}$, where the invariants $(V, \Phi)/R$, $a_0, b_0, \dots, i_0$ may be prescribed arbitrarily, up to the restrictions imposed by Lemma 3.

Proof. Consider first the (trivial) case where $b_0 = \dim V/R < \infty$. Then $d_0 = e_0 = f_0 = g_0 = h_0 = i_0 = 0$ and in $\hat{E}$ we have $\hat{V} = \hat{E}$. Let W be a space of the prescribed isometry type $(V, \dagger)/R$, and choose an embedding $R \subset H$ with $R \subset R^\perp = R^{\perp\perp}$ and $\dim R = a_0$, $\dim R^{\perp\perp}/R = c_0$ ($= 0$ if $a_0 < \infty$). Now put $E := H \overset{\perp}{\oplus} W \overset{\perp}{\oplus} \hat{E}$, $V := R + W$.

Next we deal with the case where $b_0 = \aleph_0$ but $(V, \Phi)/R$ is not hyperbolic, say $n^-(V, \Phi) < \infty$. Then we may even assume that $n^-(V, \Phi) = 0$, for we can adjust $n^-(V, \Phi)$ at the end of the construction by orthogonally adding to E the missing part V_- of V . A similar argument takes care of the indices a_0, c_0 , as in the first paragraph of this proof. Now it seems advisable to adapt the remaining indices $b_0, e_0, f_0, g_0, h_0, i_0, d_0$ step by step (in the order indicated) from zero to their prescribed values, lest we should lose control. In each case, we shall indicate how the forms are to be defined, but most of the verifications shall be left to the reader.

Step b_0 . Let (V, Φ) be positive definite, $\dim V = \aleph_0$. We extend Φ onto $E := V \oplus \hat{E}$ as follows (decompose $\hat{E} = \hat{V} \oplus Q$)

$$(i) \quad \Phi|_{\hat{V} \times \hat{E}} = \hat{\Phi}|_{\hat{V} \times \hat{E}},$$

$$(ii) \quad \Phi|_{\hat{V} \times V} = 0,$$

(iii) $\phi|_{Q \times V}$ such that $V \subset V \oplus Q$ will be dense and of finite type,

(iv) $\phi|_{Q \times Q} = \hat{\phi}|_{Q \times Q} + (\phi V \phi)|_{Q \times Q}$.

Observe that by (iii) the form $\phi V \phi$ is already defined on Q , so that

(iv) makes sense. We claim that for $E := (E, \phi, V)$ we have

$$(9) \quad E_1 = E, \quad r \phi_1 = V, \quad V^\perp = \hat{V}, \quad \phi_1|_{\hat{E} \times \hat{E}} = \hat{\phi},$$

which obviously implies $E_1 \cong \hat{E}$, $b_0 = \aleph_0$, $a_0 = c_0 = \dots = i_0 = 0$. Let us prove (9) in detail, since these verifications are typical of many which shall be omitted. First of all we have to make sure that (E, ϕ) is nondegenerate: If $x = v + \hat{v} + q \in E^\perp$ then (by (ii)) $v + q \in V^\perp$, hence (by (iii)) $v + q = 0$, so that (by (i)) $x = \hat{v} = 0$. The equality $E_1 = E$ is immediate from (ii) and (iii). From $x = v + \hat{v} + q \in V^\perp$ we infer as before that $v + q = 0$, thus showing that $V^\perp = \hat{V}$. From this and (i) we get $\phi_1|_{\hat{V} \times \hat{E}} = \phi|_{\hat{V} \times \hat{E}} = \hat{\phi}|_{\hat{V} \times \hat{E}}$. Together with (iv) this entails that $\phi_1|_{\hat{E} \times \hat{E}} = \hat{\phi}$, and this equality also implies that $r \phi_1 = V$.

Step e_0 . We orthogonally add to the embedding just constructed a par-seval embedding of codimension e_0 .

Step f_0 . We orthogonally add to the embedding obtained in the previous step a dense embedding of infinite type and codimension f_0 .

Step g_0 . By Lemma 3 ii) and iii) this step is to be left out if $(\hat{V}, \hat{\phi})/\hat{R}$ is hyperbolic or finite dimensional. Hence assume that, say, $n^-(\hat{V}, \hat{\phi}) < \infty$, $n^+(\hat{V}, \hat{\phi}) = \aleph_0$. Then we may assume w.l.o.g. that $n^-(\hat{V}, \hat{\phi}) = 0$, for $\hat{V}_-$ can orthogonally be split off. Let $(\dot{E}, \dot{\phi}, \dot{V})$ be the embedding obtained in the previous step (hence $\dot{E} = \dot{E}_1 + \dot{V}^{\perp\perp}$ and $\dot{E}_1 \cong \hat{E}$). Let G be a vector space of dimension g_0 and decompose $\dot{E} = \dot{V} \oplus \dot{V}^\perp \oplus Q$ such that $Q = U_1 \oplus U_2 \oplus U_3$ where U_1 is a supplement of $\dot{E}_1 \cap \dot{V}^{\perp\perp}$ in $\dot{V}^{\perp\perp}$,

U_2 a supplement of $\dot{V}^1 + \dot{V}^{11}$ in $\dot{F}_0$ and $U_3 \subset \dot{E}_1$, further $\dot{V}^1 = \dot{S} \oplus W$ with $\dot{S}$ the radical of $\dot{V}^1$ and W positive definite of infinite dimension (by the assumption on $(\hat{V}, \hat{\Phi})/\hat{R}$). Then we extend $\dot{\Phi}$ on $\dot{E}$ to Φ on $E := G \oplus \dot{E}$, as follows:

- $\Phi|_{G \times \dot{S}} = 0$,
- $\Phi|_{G \times \dot{V}}$ such that $\dot{V} \subset \dot{V} \oplus U_1 \oplus G$ is dense of infinite type,
- $\Phi|_{G \times W}$ such that $W \subset W \oplus U_2 \oplus G$ is dense of infinite type.

The remaining angles $(\Phi|_{G \times Q}$ and $\Phi|_{G \times G})$ may be chosen freely. It has to be shown that for $(E, \Phi, V) := (E, \Phi, \dot{V})$ we have

$$(10) \quad E_1 = \dot{E}_1, \quad V^1 = \dot{V}^1, \quad V^{11} = \dot{V}^{11}, \quad F_0 = \dot{F}_0 + G,$$

from which the desired properties of (E, Φ, V) are plain.

Step h_0 . By Lemma 3 iii) this step has to be carried out only if $\dim \hat{V}/\hat{R} = \aleph_0$. We first assume that $n^-(\hat{V}, \hat{\Phi}) < \infty$, w.l.o.g. $n^-(\hat{V}, \hat{\Phi}) = 0$. Let $(\dot{E}, \dot{\Phi}, \dot{V})$ be an embedding as obtained at the end of the preceding step; hence $\dot{E} = \dot{E}_1 + \dot{F}_0$ and $\dot{E}_1 \cong \hat{E}$. Let H be a vector space of dimension h_0 and decompose $\dot{E} = \dot{V} \oplus \dot{V}^1 \oplus Q$, $Q = U_1 \oplus U_2 \oplus U_3$ where, this time, U_1 is a supplement of $\dot{E}_1 \cap \dot{F}_0$ in $\dot{F}_0$, U_2 is a supplement of $\dot{E}_1 \cap \dot{F}_0$ in $\dot{E}_1$, and $U_3 \subset \dot{E}_1 \cap \dot{F}_0$, further $\dot{V}^1 = \dot{S} \oplus W$. We extend $\dot{\Phi}$ to Φ on $E := H \oplus \dot{E}$ in the following way:

- $\Phi|_{H \times \dot{S}} = 0$,
- $\Phi|_{H \times \dot{V}}$ such that $\dot{V} \subset \dot{V} \oplus U_1 \oplus H$ is dense of infinite type,
- $\Phi|_{H \times W}$ such that $W \subset W \oplus U_2 \oplus H$ is dense of infinite type.

Again, the remaining angles $\Phi|_{H \times Q}$ and $\Phi|_{H \times H}$ may be defined arbitrarily. Here we can show that $(E, \Phi, V) := (E, \Phi, \dot{V})$ satisfies

$$(11) \quad E_1 = \dot{E}_1, \quad V^1 = \dot{V}^1, \quad F_0 = \dot{F}_0, \quad V^{11} = \dot{V}^{11}, \quad S^1 = \dot{S}^1 + H$$

which gives what we want in case $n^-(\hat{V}, \hat{\Phi}) < \infty$. When $(\hat{V}, \hat{\Phi})/\hat{R} \cong (W, \hat{\Phi})$ is hyperbolic we replace the third condition in the definition of Φ by

$$- \quad \Phi|_{H \times W} \text{ such that } W \subset W \oplus U_2 \oplus H \text{ is dense.}$$

It is true that our considerations preceding the theorem do not apply here, but for μ -dense hyperbolic embeddings the situation is even simpler: They may be constructed in the form $E = V \oplus Q$, $Q \subset V^\times$, $Q \cap \mu V = 0$, with $\Phi|_{V \times Q}$ the canonical form and $\Phi|_{Q \times Q}$ arbitrary. (μ is the map $V \rightarrow V^\times$, $v \mapsto \Phi(-, v)$.) For the verification of (11) in the case of a hyperbolic W one has to observe that $\dot{F}_O = \dot{V}^i + \dot{V}^{ii}$.

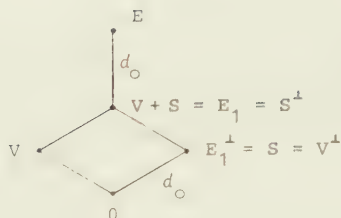
Step i_O . Again, we denote by $(\dot{E}, \dot{\Phi}, \dot{V})$ the embedding constructed in the last step, hence $\dot{E}_1 \cong \hat{E}$ and $\dot{E} = \dot{E}_1 + \dot{S}^i$. By Lemma 3 i) we may assume that $\dim \hat{R} = \dim \dot{S} = \aleph_0$, otherwise $i_O = 0$ and this step could be omitted. Let I be a vector space of dimension i_O and extend $\dot{\Phi}$ to Φ on $E := I \oplus \dot{E}$ in the following way

$$\begin{aligned} - \quad & \Phi|_{I \times \dot{V}} \text{ such that } \dot{V} \subset \dot{V} \oplus U \oplus I \text{ is dense of infinite type,} \\ & \text{where } U \text{ is a supplement of } \dot{E}_1 \text{ in } \dot{E}, \\ - \quad & \Phi|_{I \times \dot{S}} \text{ such that } \dot{S}^i \subset \dot{E}. \end{aligned}$$

The second condition can be fulfilled because $\dim \dot{S} = \aleph_0$. The reader may verify that $(E, \Phi, V) := (E, \Phi, \dot{V})$ satisfies

$$(12) \quad E_1 = \dot{E}_1, \quad V^i = \dot{V}^i, \quad F_O = \dot{F}_O, \quad V^{ii} = \dot{V}^{ii}, \quad S^i = \dot{S}^i, \quad E_1^{ii} = E = \dot{E}_1^{ii} + I.$$

Step d_O . The easiest way to adapt the index d_O is to orthogonally add to the embedding constructed in the preceding step an embedding (E, Φ, V) with V positive definite, whose lattice $V(E)$ has the shape



To obtain such an embedding, put $E := V \oplus S \oplus U$ with $\dim S = \dim U = d_0$ and define Φ as follows:

- $\Phi|_{V \times V}$ positive definite,
- $\Phi|_{V \times S} = 0$,
- $\Phi|_{V \times U}$ such that $V \subset V \oplus U$ is dense of infinite type,
- $\Phi|_{(S+U) \times (S+U)}$ hyperbolic with $S \subset S^\perp$, $U \subset U^\perp$.

Thereby the proof of Theorem 2 is complete in the case $n^-(V, \Phi) < \infty$. If $(V, \Phi)/R$ is to be hyperbolic then $e_0 = i_0 = 0$ and $\hat{V} = \hat{E}$, by Lemma 3. We may now repeat the above steps with most of the arguments greatly simplified (steps e_0 and i_0 drop out). E.g., step b_0 consists in orthogonally adding to $\hat{E}$ a hyperbolic space (V, Φ) , and in step g_0 the condition on $\Phi|_{G \times \dot{V}}$ must be replaced by

- $\Phi|_{G \times \dot{V}}$ such that $\dot{V} \subset \dot{V} \oplus U_1 \oplus G$ is dense.

This concludes the proof of Theorem 2.

Corollary 1. There exist admissible embeddings f with $V_\infty = E_\infty$ and arbitrarily prescribed values of the indices $a_j, b_j, \dots, i_j$ ($j \in \mathbb{N}$) and isometry types of $(V_j, \Phi_j)/R_j$ ($j \in \mathbb{N}$), provided that the conditions of Lemma 3 are satisfied.

Proof. From the theorem it is evident that embeddings with "arbitrarily" prescribed finite lattices $V(E)$ can be constructed. Embeddings E with infinite lattices are obtained by forming orthogonal sums $\bigoplus_{1 \in I} E_1$

(with $\text{card } (I) \leq \aleph_0$). However, one has to take care that the indices behave additively. From the definition of $V(E)$ in II.1 it is evident that it suffices to make sure that the spaces E_i and F_i ($i \in \mathbb{N}$) behave additively. This in turn is warranted if we choose the summands E_i in such a way that for each $j \in \mathbb{N}$ we have

$$(13) \quad \begin{aligned} &\text{if } \bigoplus_{i \in I} (V_{ij}, \phi_{ij}) / R_{ij} \text{ is hyperbolic then each } (V_{ij}, \phi_{ij}) / R_{ij} \\ &\text{is finite dimensional or hyperbolic.} \end{aligned}$$

From this proof and the Congruence Theorem we deduce

Corollary 2. Every admissible embedding E with zero top admits an orthogonal decomposition $E = \bigoplus_{i \in I} E_i$ such that each E_i has finite lattice $V(E_i)$.

4.3 Attaching the top. Lemma 4 describes a connection of the top with the invariants (B) of an embedding (cf. 2.1). We are going to show that there are no other relations.

Theorem 3. Let E be an embedding (admissible or not) with infinite lattice $V(E)$ and zero top, and let L be a vector space of countable dimension endowed with a sequence $(\Omega_i)_{i \in \mathbb{N}}$ of hermitean forms such that condition (8) holds. Then ϕ can be extended to a form ϕ' on $E' := L \oplus E$ in such a way that $E' := (E', \phi', V)$ has the same invariants (A) and (B) as E and its top has $(L, (\Omega_i)_{i \in \mathbb{N}})$ as a lifting. If E is admissible then so is E' .

Proof. The extension ϕ' of ϕ will be chosen such as to satisfy for each $i \in \mathbb{N}$

$$(14) \quad \begin{aligned} E'_i &= L \oplus E_i, \quad r \phi'_i = r \phi_i, \quad V'^{i-1}_{i-1} = V^{i-1}_{i-1}, \quad V'_i = V_i, \\ V'^{i-1}_{i-1} V'^{i-1}_{i-1} &= V^{i-1}_{i-1} V^{i-1}_{i-1}, \quad \phi'_i|_{L \times L} = \Omega_i. \end{aligned}$$

(The shift of indices in two of these equations has technical reasons.) From this the assertions of the theorem are easy to obtain: We inductively infer that $\phi'_i|_{E_i \times E_i} = \phi_i$ so that the invariants (B) of E' coincide with those of E . Furthermore, $E'_\infty = L \oplus E_\infty = L \oplus V_\infty$ and $V'_\infty = V_\infty$ so that in view of the last equation of (14) the top of E' turns out to have the prescribed shape. Finally, from (14) we can also conclude that if $D \in V(E)$ then the corresponding space in $V(E')$ is either D or $L \oplus D$ (depending on whether $D \subset V_\infty + Z$ or $D \supset V_\infty = E_\infty$), so that the indices (A) of E' are the same as those of E .

Let us now turn to the construction of ϕ' . We start by choosing, for each $i \in \mathbb{N}$, a fundamental decomposition of (V_i, ϕ_i) ,

$$V_i = R_i \oplus W_i \oplus U_i$$

such that $\dim W_i = \min \{n^+(V_i, \phi_i), n^-(V_i, \phi_i)\} < \infty$ and $\dim U_i = \kappa_0$ (Lemma 3). Our aim is to extend the forms $\phi_i|_{V_\infty \times V_\infty}$ to certain forms $\hat{\phi}_i$ on $L \oplus V_\infty$. We shall not specify the values for $\hat{\phi}_i|_{L \times L}$, hence it is enough to define $\hat{\phi}_i|_{L \times V_\infty}$. We first choose

- (i) $\hat{\phi}_i|_{L \times R_i} = 0$,
- (ii) $\hat{\phi}_i|_{L \times (W_i + U_i)}$ such that $(\hat{\phi}_i(W_i + U_i)\hat{\phi}_i)|_{L \times L} = \Omega_i - \Omega_{i-1}$ and such that the functionals $\hat{\phi}_i(-, l)$ on $W_i + U_i$ (for $l \in L \setminus \{0\}$) are linearly independent of $\{\phi_i(-, e) \mid e \in E_i\}$.

The reason why (ii) can be fulfilled is condition (8). Indeed, choose a basis (l_j) of L in which $\Omega_i - \Omega_{i+1}$ is diagonal and decompose $\Omega_i - \Omega_{i+1} = \chi_u + \chi_w$ where χ_u and χ_w are likewise diagonal, χ_u definite, χ_w semidefinite of rank $\leq \dim W_i$ (which is made feasible by (8)). Now it is possible to choose the values of $\hat{\phi}_i(l_j, u)$, $\hat{\phi}_i(l_j, w)$ for $u \in U_i$, $w \in W_i$ in such a way that $(\hat{\phi}_i U_i \hat{\phi}_i)|_{L \times L} = \chi_u$, $(\hat{\phi}_i W_i \hat{\phi}_i)|_{L \times L} = \chi_w$. In doing so it is easy to obey the requirement concerning linear inde-

pendence (cf. the considerations at the beginning of 4.2).

The next step is to extend the $\hat{\phi}_i|_{L \times V_i}$ onto $L \times V_{i+1}$ (simultaneously for all $i \in \mathbb{N}$), and then onto $L \times V_{i+2}$, $L \times V_{i+3}$, ..., hence finally onto $L \times V_\infty$, and this we do (recursively) by solving for $\hat{\phi}_i|_{L \times V_j}$ from the equations (where $0 \leq i \leq j$)

$$(iv) \quad \hat{\phi}_{i+1}|_{L \times V_j} = \hat{\phi}_i|_{L \times V_j} - (\hat{\phi}_i(W_i + U_i)\hat{\phi}_i)|_{L \times V_j}.$$

It is readily verified that the $\hat{\phi}_i|_{L \times V_j}$ are defined unambiguously, and from (iv) we get

$$(v) \quad \hat{\phi}_{i+1}|_{L \times V_\infty} = \hat{\phi}_i|_{L \times V_\infty} - (\hat{\phi}_i(W_i + U_i)\hat{\phi}_i)|_{L \times V_\infty}.$$

Now let U be any supplement of V_∞ in E and define ϕ' by

$$\phi'|_{E \times E} := \phi, \quad \phi'|_{L \times V_\infty} := \hat{\phi}_0|_{L \times V_\infty}, \quad \phi'|_{L \times U} := 0, \quad \phi'|_{L \times L} := \Omega_0.$$

We shall show by induction on i that (14) and the following assertions hold:

$$(15) \quad \phi'_i|_{E_i \times E_i} = \phi_i, \quad \phi'_i|_{L \times V_\infty} = \hat{\phi}_i|_{L \times V_\infty}.$$

For $i = 0$ everything is true by definition, except for $r\phi' = r\phi = 0$, which is a consequence of (ii): If $l + e$ with $l \in L$, $e \in E$ is in $r\phi'_1$ then $\hat{\phi}_0(-, l) = -\phi_0(-, e)$ on $W_0 + U_0$ which by (ii) implies $l = 0$. Assume then that (14) and (15) hold for $i \leq n$. Hence $E'_{n+1} = C(E'_n, \phi'_n, V'_n) = C(L \oplus E_n, \phi'_n, V_n) = L \oplus E_{n+1}$ by (i) and (ii), and the equations (15) for $i = n+1$ follow from the inductive assumptions and (v). The second and third equation in (14) again follow from the condition on linear independence in (ii), and the fourth equation is a consequence of these two. As to the fifth equation, note that $W_{n+1} + U_{n+1}$ can be chosen to be a supplement of S_n in $V_n^{\perp n}$, and $\hat{\phi}_n|_{L \times V_n^{\perp n}} = \hat{\phi}_{n+1}|_{L \times V_n^{\perp n}}$ by (v), so that a further application of (ii) yields the equation. Finally,

$$\phi'_{n+1}|_{L \times L} = \phi'_n|_{L \times L} - (\phi'_n(W_n + U_n)\phi'_n)|_{L \times L} = \Omega_n - (\Omega_n - \Omega_{n+1}) = \Omega_{n+1},$$

by (ii).

This theorem can be combined with the Congruence Theorem and Corollary 1 to Theorem 2 to give the following results.

Corollary 1. There exist admissible embeddings with arbitrarily prescribed invariants (A), (B), (C) (as defined in 2.1), provided that these invariants conform to the restrictions expressed in Lemma 3 and Lemma 4.

Corollary 2. Each admissible embedding E with nonzero top admits an orthogonal decomposition $E = E' \overset{1}{\oplus} E''$ with the following properties: E' has the same top as E and the same invariants (B), but its indices (A) vanish (except for $b'_1 = \aleph_0$); E'' has zero top, definite forms (B) and the same indices (A) as E .

5. Supplementary remarks on $V(E)$

5.1 We have mentioned at the beginning of Section 1 that (1), the requirement that E be admissible, actually entails the complete distributivity of $\tilde{V}(E)$, the completion of $V(E)$ in $L(E)$. Let us show how this can be derived from the results of the preceding section. Nothing has to be shown if $V(E)$ is finite. If $V(E)$ is infinite we may decompose $E = E' \overset{1}{\oplus} E''$ as in Corollary 2 above ($E' = 0$ if E has zero top), and $E'' = \overset{1}{\oplus}_{i \in I} E_i$ where each E_i has finite lattice (Corollary 2 to Theorem 2). Since (13) is fulfilled the elements of $V(E)$ behave additively with respect to this decomposition, and we may therefore interpret $V(E)$ as a sublattice of $V(E') \times \prod_{i \in I} V(E_i) \subset L(E)$. But $V(E')$ and each $V(E_i)$ are complete so that the product lattice $V(E') \times \prod V(E_i)$ is likewise complete and hence contains $\tilde{V}(E)$. Furthermore $V(E')$ and the $V(E_i)$ are completely distributive, whence so are $V(E') \times \prod V(E_i)$ and $\tilde{V}(E)$.

Using the complete distributivity along with results of Chap. III it is easy to see that the set $\{A + \sum_{j \in I} V_j^{1j1j} \mid A \in V_{\text{red}}(E), I \subset \mathbb{N}\}$ is a complete sublattice of $L(E)$; hence we have

Proposition 2. If E is admissible then $\tilde{V}(E)$ is completely distributive and equals the set $\{A + \sum_{j \in I} V_j^{1j1j} \mid A \in V_{\text{red}}(E), I \subset \mathbb{N}\}$.

5.2 In the light of the Congruence Theorem it appears very plausible that if E has zero top then $V(E)$ must contain along with each $W \in V(E)$ the spaces $C(W)$ and $r\phi_W$. In fact it is possible to determine the assignments $V(E) \rightarrow L(E), W \mapsto C(W), r\phi_W$ for E an arbitrary embedding in dependence of the invariants $(A), (B)$ and (C) . This we shall do now.

Let L be a supplement of V_∞ in E_∞ and put $\Omega_i = \phi_i|_{L \times L}$. Assume for the moment that $(V_\infty, \phi)/S$ is not hyperbolic, say $n^+(V_\infty, \phi) = \aleph_0 > n^-(V_\infty, \phi)$. Since $n^-(V_\infty, \phi) = \sum_{i \in \mathbb{N}} n^-(V_i, \phi_i)$ by I.2.(3) we see that $n^-(V_i, \phi_i) = 0$ for large i , whence $\Omega_i - \Omega_{i+1}$ is positive definite for large i , by Lemma 4. Applying Schwarz' inequality to the forms $\Omega_i - \Omega_{i+1}$ we get that the set

$$L_\infty := \{x \in L \mid \lim_{i \rightarrow \infty} \Omega_i(x, x) \text{ exists}\}$$

is a subspace of L and the formula

$$\Omega_\infty(x, y) := \lim_{i \rightarrow \infty} \Omega_i(x, y)$$

defines a form Ω_∞ on L_∞ .

Proposition 3. Let E be an arbitrary embedding and $W \in V(E)$ such that $(W, \phi)/W \cap V_\infty^\perp$ is not hyperbolic.

- If $W \supset E_\infty$ then $C(W) = r\phi_W = W$.
- If $W = V_\infty$ then $C(W) = V_\infty \oplus L_\infty$ and $\phi_W = 0 \oplus \Omega_\infty$.
- If $W \subset V_\infty$ then $C(W) = C(E_i, \phi_i, W)$ and $\phi_W = (\phi_i)_W$ for each i with $r\phi_i \subset W$.
- If $W \subset V_\infty + Z$ then $C(W) = W + C(W \cap V_\infty)$ and $\phi_W(w+x, w'+x') = \phi_W \cap V_\infty(x, x')$ for $w, w' \in W, x, x' \in C(W \cap V_\infty)$.

To complete the picture we have to determine $(C(W), \phi_W)$ for $W \in V(E) \setminus [r\phi_1, E] = [0, V + V^\perp]$. This is easy: If $W \in [0, S]$ then $(C(W), \phi_W) = (W^\perp, \phi)$ and $(C(V+W), \phi_{V+W}) = (E_1 \cap W^\perp, \phi_1)$; furthermore $(C(V^\perp), \phi_{V^\perp}) = (F, \psi)$, and finally $(C(V + V^\perp), \phi_{V+V^\perp}) = (E_2, \phi_2)$ if $(V + V^\perp, \phi)/S$ is not hyperbolic.

We omit the proof of Proposition 3 because we do not need these facts in the sequel. It can be supplied by making systematic use of the relations (5) and (6) in I.2.2 (whose proof was left out too).

6. A remark on non-admissible embeddings

So far nothing has been said about the existence of non-admissible embeddings. We shall show by construction that non-admissible embeddings with "arbitrarily" prescribed invariants (A), (B), (C) (cf. Section 2) and a further invariant (D) (to be defined in a moment) exist. However, there seems to be no evidence for this enlarged set of invariants to characterize the congruence class of an embedding; besides, the new invariant is of a rather messy kind.

Let E be an arbitrary embedding and recall the notations

$$Z_n := V_0^{\perp 0} + \dots + V_{n-1}^{\perp n-1} \quad (\text{for } n \geq 1), \quad Z := \bigcup_{n \geq 1} Z_n. \text{ Set}$$

$$Q := \bigcap_{n \in \mathbb{N}} (E_n + Z_n) = \bigcap_{n \in \mathbb{N}} (E_n + Z), \quad Y := Q / E_\infty + Z,$$

$$Y_n := V_n^{\perp n} / V_{n+1}^{\perp n+1}.$$

Each $x \in Q$ may be written as $x = x_0 + \dots + x_{n-1} + e_n$ with $x_i \in V_i^{\perp i+1}$ and $e_n \in E_n$. The cosets $x_i + V_{i+1}^{\perp i+1}$ are uniquely determined by x and do not depend on the choice of $n > i$; hence for each i we obtain a linear map

$$\pi_i : Q \longrightarrow Y_i, \quad x \longmapsto x_i + V_{i+1}^{\perp i+1}.$$

The kernel of $\pi := (\pi_i)_{i \in \mathbb{N}} : Q \longrightarrow \prod_{i \in \mathbb{N}} Y_i$ turns out to be E_∞ , and the image of $E_\infty + Z$ is $\bigoplus_{i \in \mathbb{N}} Y_i$. There results an injective linear map

$$\bar{\pi} : Y \longrightarrow \prod Y_i / \bigoplus Y_i.$$

If $E \equiv E'$ is a congruence then the two maps $\bar{\pi}, \bar{\pi}'$ are isomorphic in the following sense: There is a family of isomorphisms $Y_i \xrightarrow{\sim} Y'_i$ such that the induced isomorphism $\prod Y_i / \bigoplus Y_i \xrightarrow{\sim} \prod Y'_i / \bigoplus Y'_i$ maps $\bar{\pi}(Y)$ onto $\bar{\pi}'(Y')$. Our new invariant is

$$(D) \quad \text{the isomorphism class of } \pi : Y \hookrightarrow \prod Y_i / \bigoplus Y_i.$$

We observe that if E is admissible then $Y = 0$ so that (D) is determined by (A) since $\dim Y_i = \delta_i$. We may now state

Theorem 4. Let E be an admissible embedding such that infinitely many of the spaces $Y_i := V_i^{i+1} / V_{i+1}^{i+1}$ are $\neq 0$. Let Y be a space of countable dimension and $\bar{\pi} : Y \longrightarrow \prod Y_i / \bigoplus Y_i$ an injective linear map. Then Φ can be extended to a form Φ' on $E' := Y \oplus E$ in such a way that $E' := (E', \Phi', V)$ has the same invariants (A), (B), (C) as E , and $\bar{\pi}' = \bar{\pi}$.

Proof. We shall use arguments similar to those in the proof of Theorem 3 in 4.3. In order to simplify the problem we remark that if $E = E_1 \oplus E_2$ such that the indices $\delta_i = \dim Y_i$ are realized wholly within E_1 we may forget about E_2 and assume $E = E_1$. By the results of Section 4 there is such a decomposition where the only nonzero indices (A) in E_1 are the $b_i = \aleph_0$ and the δ_i and where E_1 has definite invariants (B) and zero top (C). Thus we may assume that $E = Q = V_\infty + Z$ and $V_{i+1} = V_i + V_i^{i+1}$. Hence if we choose supplements U_i of R_i in V_i and supplements X_i of V_{i+1}^{i+1} in V_i^{i+1} then $E_n = \bigoplus_{i \in \mathbb{N}} U_i \oplus \bigoplus_{i \geq n} X_i$

and $\bar{\pi}$ admits a "lifting" to a linear map $\pi = (\pi_i)_{i \in \mathbb{N}} : Y \longrightarrow \prod_{i \in \mathbb{N}} X_i$. We set $Y(i) := \{Y - \pi_0(Y) - \dots - \pi_{i-1}(Y) \mid Y \in Y\} \subset E' = Y \oplus E$ for $i \geq 1$, $Y(0) := Y$. Our strategy is to extend the forms $\hat{\phi}_n$ on E_n simultaneously to certain forms $\hat{\phi}_n$ on $Y(n) \oplus E_n$ and then to show that for $V' := V$ and $\hat{\phi}' := \hat{\phi}_0$ we have $\hat{\phi}'_i = \hat{\phi}_i$. Note that for $i \geq n$ we have $Y(n) \oplus E_n = Y(i) \oplus E_n$; therefore specifying the values of, say, $\hat{\phi}_n|_{U_i \times Y(n)}$ for $i \geq n$ is equivalent to specifying the values of $\hat{\phi}_n|_{U_i \times Y(i+1)}$. Now define

- (i) $\hat{\phi}_n|_{U_i \times Y(n)} := 0$ for $i < n$
- (ii) $\hat{\phi}_n|_{U_n \times Y(n+1)}$ such that $(\hat{\phi}_n \vee_n \hat{\phi}_n)|_{Y(n+1) \times Y(n+1)}$ exists and the functionals $\hat{\phi}_n(-, y)$ on U_n (for $y \in Y(n+1) \setminus \{0\}$) are linearly independent of $\{\hat{\phi}_n(-, x) \mid x \in E_n\}$.

This takes care of $\hat{\phi}_n|_{V_n \times Y(n)}$ and will imply that $Y(n+1) \oplus E_{n+1} = C(Y(n) \oplus E_n, \hat{\phi}_n, V_n)$. In particular we may infer that $(\hat{\phi}_n \vee_n \hat{\phi}_n)|_{U_j \times Y(j)}$ exists for $j > n$. Hence we can determine the $\hat{\phi}_n|_{U_j \times Y(j)}$ (and thus $\hat{\phi}_n|_{U_j \times Y(n)}$) by recursion upon $j - n$ from the equations

$$(iii) \quad \hat{\phi}_{n+1}|_{U_j \times Y(j)} = \hat{\phi}_n|_{U_j \times Y(j)} - (\hat{\phi}_n \vee_n \hat{\phi}_n)|_{U_j \times Y(j)}.$$

Similarly, since $(\hat{\phi}_n \vee_n \hat{\phi}_n)|_{X_j \times Y(j)}$ exists for $j > n$ we can solve for $\hat{\phi}_n|_{X_j \times Y(j)}$ (and hence for $\hat{\phi}_n|_{X_j \times Y(n)}$) from

$$(iv) \quad \hat{\phi}_{n+1}|_{X_j \times Y(j)} = \hat{\phi}_n|_{X_j \times Y(j)} - (\hat{\phi}_n \vee_n \hat{\phi}_n)|_{X_j \times Y(j)}$$

after having fixed the values of $\hat{\phi}_n|_{X_n \times Y(n)}$ in an arbitrary fashion. So far we have defined $\hat{\phi}_n|_{E_n \times Y(n)}$. In order to specify $\hat{\phi}_n|_{Y(n) \times Y(n)}$ choose $\hat{\phi}_0|_{Y(0) \times Y(0)}$ arbitrarily and put

$$(v) \quad \hat{\phi}_{n+1}|_{Y(n+1) \times Y(n+1)} = \hat{\phi}_n|_{Y(n+1) \times Y(n+1)} - (\hat{\phi}_n \vee_n \hat{\phi}_n)|_{Y(n+1) \times Y(n+1)}.$$

This terminates the construction of the extensions $\hat{\phi}_n$ of ϕ_n . We have already remarked that $C(Y(n) \oplus E_n, \hat{\phi}_n, V_n) = Y(n+1) \oplus E_{n+1}$; furthermore it is obvious from the construction that $(\hat{\phi}_n)_{V_n} = \hat{\phi}_{n+1}$ and from (i) and (ii) it follows that $r_{\hat{\phi}_n} = r_{\phi_n}$ ($= V_{n-1}$ for $n \geq 1$). To finish the proof we put $\phi' := \hat{\phi}_0$ and $V' := V$; by induction on n one shows that

$$(16) \quad V_n'^{\perp n} = V_n^{\perp n}, \quad V_{n+1}' = V_{n+1}, \quad \phi_{n+1}' = \hat{\phi}_{n+1}, \quad V_n'^{\perp n \perp n} = V_n^{\perp n \perp n}$$

(this is left to the reader; cf. the proof of Theorem 3 after (15)).

Hence we get $E_n' = Y(n) \oplus E_n$, $E_\infty' = \bigcap E_n' = \bigcap E_n = E_\infty = V_\infty$, $E_n' + Z_n' = Y(n) \oplus (E_n + Z_n) = Y \oplus (E_n + Z_n) = E'$ (since $Y \equiv Y(n)$ modulo Z_n) and therefore $Q' = \bigcap (E_n' + Z_n') = E'$, $E_\infty' + Z' = E_\infty + Z = E$. The assertions of the theorem are now evident.

CHAPTER V

EXAMPLES AND APPLICATIONS

The results of the last chapter permit to completely classify the admissible embeddings with zero top, in particular, all embeddings E with finite lattice $V(E)$. This is especially useful because the finiteness of $V(E)$ can frequently be derived from elementary and naturally occurring properties of E . The simplest example of this kind is furnished by the 1-dense embeddings; these have been discussed in I.4. We shall now pick out some more cases of interest.

1. Orthogonally closed embeddings

From Figure 5 in IV.4 we read off that E is 1-closed (i.e. $V = V^{11}$) if and only if

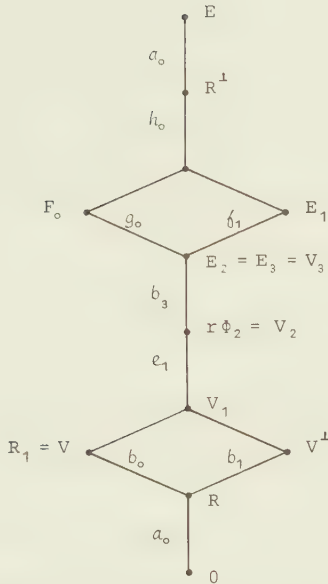
$$c_0 + d_0 + e_0 + a_1 + c_1 + d_1 + a_2 + b_2 + f_0 = 0$$

which by IV.4 Lemma 3 is equivalent to saying that at most the following indices differ from zero:

$$a_0, b_0, b_1, b_3, e_1, f_1, g_0, h_0.$$

Thereby $V(E)$ assumes the shape shown in Figure 6 below. Hence the Congruence Theorem tells us that the orbit of a 1-closed subspace $V \subset (E, \phi)$ under the action of the orthogonal group of (E, ϕ) is characterized, within the set of orbits of closed subspaces, by the following data:

- the isometry types of $(V, \phi)/R$, $(V^\perp, \phi)/R$ ($\cong (V_1, \phi_1)/R_1$) and $(E_2, \phi_2)/r\phi_2$ where $\phi_2 = \phi - \phi V \phi - \phi V^\perp \phi$ is defined on $E_2 = C(V) \cap C(V^\perp)$,
- the dimensions of $R, r\phi_2/V + V^\perp, C(V)/E_2, C(V^\perp)/E_2$ and $R^\perp/C(V) + C(V^\perp)$.

Figure 6 : $V(E)$ if $V = V^{\perp\perp}$

Corollary 1 . Each $\perp$ -closed embedding E admits an orthogonal decomposition

$$(1) \quad (E, V) = (R \oplus R', R) \oplus (H_0, H_0) \oplus (H_1, 0) \oplus \bigoplus_{i \in I} (E_i, V_i)$$

such that

- $R = V \cap V^{\perp}$, $R' \subset R'^{\perp}$
- each V_i is nondegenerate, $\dim V_i = \kappa_0 = \dim V_i^{\perp}$, $\dim E_i/V_i + V_i^{\perp} = 1$,
- if V/R is hyperbolic then each V_i is hyperbolic , otherwise each V_i is definite,
- if $V^{\perp}/R$ is hyperbolic then each $V_i^{\perp}$ is hyperbolic , otherwise each $V_i^{\perp}$ is definite.

This result and the corresponding on dense embeddings (Corollary to I.4 Theorem 1) mean that $\perp$ -dense and $\perp$ -closed embeddings split in the sense of [12] p.316 (the first summand in (1) vanishes if ϕ is

definite). Thus Corollary 1 is an improvement of the corollary on p.319 loc.cit.. The first three summands in (1) may be regarded as "trivial"; hence consider now a closed nondegenerate embedding E with $\dim V = \aleph_0 = \dim V^\perp$, $\dim E/V + V^\perp = 1$, where V and $V^\perp$ are definite or hyperbolic. Here we find that there are 33 congruence classes of such embeddings (6 where $V, V^\perp$ are positive definite, 2 where V is positive definite and $V^\perp$ hyperbolic, 1 where V and $V^\perp$ are hyperbolic). In 21 cases (E, Φ) turns out hyperbolic, in 4 cases definite, and in 8 cases we have $\min\{n^+(E), n^-(E)\} = 1$.

If $V \subset (E, \Phi)$ is nondegenerate and $\perp$ -closed then $V + V^\perp$ is $\perp$ -dense. Conversely, each $\perp$ -dense subspace $W \subset (E, \Phi)$ splits as $W = V \oplus U$ such that $V^\perp = U$ and $U^\perp = V$ in E . This is true in $\aleph_0$ -dimensional spaces over arbitrary base fields, as can be shown by an easy combinatorial argument. But for the spaces which are of interest here we can obtain much more precise information:

Corollary 2. Let $W \subset (E, \Phi)$ be a $\perp$ -dense subspace and decompose $W = \bar{V} \oplus \bar{U}$ with $\dim \bar{V} = \aleph_0 = \dim \bar{U}$. Choose subspaces $A \supset W$ ($A = W$ if $\bar{V}$ is hyperbolic) and $B \supset W$ ($B = W$ if $\bar{U}$ is hyperbolic) such that if W is not hyperbolic then $A \cap B = C(W)$. Then there is a decomposition $W = V \oplus U$ such that $V \cong \bar{V}$, $U \cong \bar{U}$, $V^\perp = U$, $U^\perp = V$ in E , $C(V) = A$, $C(U) = B$.

Proof. There exists a $\perp$ -closed nondegenerate embedding (E', Φ', V') such that with $W' := V' + V'^\perp$, $A' := C(V')$, $B' := C(V'^\perp)$ we have $V' \cong \bar{V}$, $V'^\perp \cong \bar{U}$, $\dim r\phi'_{W'}/W' = \dim r\phi_{W/W}$, $(C(W'), \phi'_{W'})/r\phi'_{W'} \cong (C(W), \phi_W)/r\phi_W$ (recall that $\phi'_{W'} = \phi'_2$ if W' is not hyperbolic, and $C(W') = W'$ otherwise), $\dim A' \cap B'/W' = \dim A \cap B/W$, $\dim A'/A' \cap B' = \dim A/A \cap B$, $\dim B'/A' \cap B' = \dim B/A \cap B$, $\dim E'/A' + B' = \dim E/A + B$. Therefore we have an obvious lattice isomorphism $\tau : \{0, W, r\phi_W, C(W), A \cap B, A, B, A+B, E\} \longrightarrow \{0', W', r\phi'_{W'}, C(W'), A' \cap B', A', B', A'+B', E'\}$ which respects indices

and commutes with ι . The corollary will follow if we can show that τ is induced by an isometry $E \rightarrow E'$. Needless to say, we use the lattice method to construct such an isometry: Choose G to be the set of all $(T: X \rightarrow X') \in F$ (notations of I.3.3) which on $X \cap C(W)/X \cap r\phi_W$ coincide with a fixed isometry $(C(W), \phi_W)/r\phi_W \xrightarrow{\sim} (C(W'), \phi_{W'})/r\phi_{W'}$. The verification of (PP) is routine (in case $D=W$ one has to make use of II.1 Lemma 1).

2. Maximal positive and positive definite subspaces

2.1 If $E = E_+ \overset{1}{\oplus} E_-$ is a fundamental decomposition then E_+ is maximal both in the set of positive definite subspaces and in the set of positive (= positive semidefinite) subspaces. But, of course, these sets also contain maximal elements which do not arise in this manner (e.g., certain dense subspaces). In the case $n^-(E, \phi) < \infty$ we are able to give a complete classification (modulo congruence) of the maximal positive (or positive definite) subspaces of (E, ϕ) . We begin by a useful characterization (here and in the first two corollaries (E, ϕ) is still arbitrary).

Proposition. For $V \subset (E, \phi)$ positive definite (positive) the following are equivalent:

- (i) V is maximal positive definite (maximal positive)
- (ii) $r\phi_1 = V + S$ ($r\phi_1 = V$) and $(E_1, \phi_1)/r\phi_1$ is negative definite.

Proof. We prove the assertions concerning positive definite subspaces; for positive subspaces the arguments run similarly.

(i) $\Rightarrow$ (ii). Let $x \in E_1$ with $\phi_1(x, x) \geq 0$. Hence for all $v \in V$ we have $\phi(x+v, x+v) \geq \phi_1(x+v, x+v) = \phi_1(x, x) \geq 0$. By the maximality of V there is $v \in V$ such that $\phi(x+v, x+v) = 0$ and $x+v \in V^\perp$ (in particular we have $r\phi_1 \subset V + V^\perp$). Thus $\phi_1(x, x) = 0$ which shows that $\phi_1/r\phi_1$ is negative,

hence negative definite. At the same time we obtain $r\phi_1 = r\phi_1 \cap (V+V^\perp) = V + r\phi_1 \cap V^\perp = V + E_1^\perp \subset V+S$. But $S \subset r\phi_1$ because $\phi_1|_{S \times S} = \phi|_{S \times S} = 0$ and ϕ_1 is negative. Hence $r\phi_1 = V+S$.

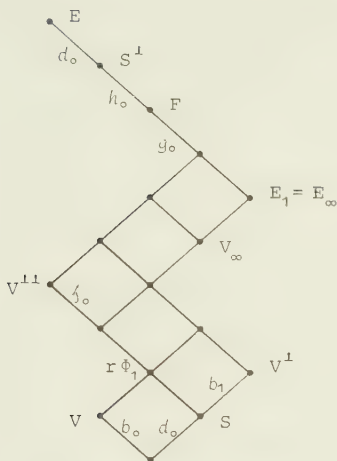
(ii) $\Rightarrow$ (i). Let $V+Kx$ be positive definite. Then $x \in E_1$ and $\phi_1(x, x) \geq 0$, hence $x \in r\phi_1 = V+S$ by (ii). The definiteness of $V+Kx$ now forces $x \in V$. Hence V is maximal.

Corollary 1. Each maximal positive definite (or maximal positive) subspace V of (E, ϕ) is admissible. If V is maximal positive definite then $n^-(E, \phi) = \dim E/V+S$; if V is maximal positive then $n^-(E, \phi) = \dim E/V$.

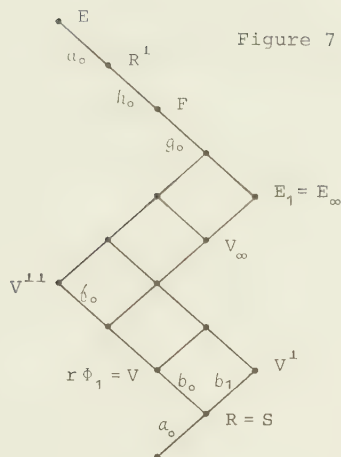
Proof. Admissibility follows from IV.1 Proposition 1 and Lemma 1, and I.2 (3) gives $n^-(E, \phi) = \dim E/r\phi_1$.

Corollary 2. If V is maximal positive definite then $V+S$ is maximal positive. Conversely, if V' is maximal positive then there is a supplement V of R' in V' which is maximal positive definite (it is uniquely determined up to congruence in (E, ϕ)).

Proof. Let V be maximal positive definite. Since $E_1 \subset S^\perp$ we have $C(V+S) = C(V) = E_1$ and $\phi_{V+S} = \phi_V$, hence by the proposition $V+S$ is maximal positive. For the reverse direction it would be sufficient (by similar reasons) to exhibit a supplement V of R' in V' with $C(V) \subset R'^\perp$. This would require a somewhat tedious recursive construction; instead we may apply the results of the last chapter: First compare the lattices $V_1(E)$ for V a maximal positive definite subspace on the one hand and V a maximal positive subspace on the other hand (Figure 7). We see that by IV.4 Theorem 2 and the Congruence Theorem there is $V \subset E$ such that $(E, \phi, V+S) \cong (E, \phi, V')$ (in the cited theorem, choose $\hat{E} = (E_1', \phi_1', V_1')/r\phi_1'$, $a_0 = 0$, $b_0 = \kappa_0$, $c_0 = 0$, $d_0 = a_0'$, $e_0 = 0$,



$V_1(E)$ when V is
maximal positive definite



$V_1(E)$ when V is
maximal positive

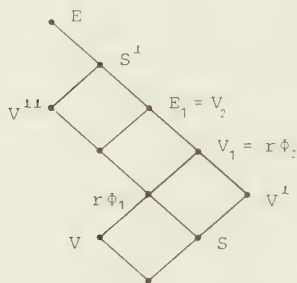
$h_0 = h'_0$, $g_0 = g'_0$, $h_0 = h'_0$, $i_0 = 0$ and (V, Φ) positive definite).

By the proposition V is maximal positive definite, and the "uniqueness" of V follows from the Congruence Theorem, since the invariants of V are completely determined by those of V' . Q.E.D.

2.2 Let us now have a look at the classification problem. If the overspace (E, Φ) is hyperbolic we cannot say much more than in the case of arbitrary admissible embeddings: The only restriction on the embedding $E_1 = (E_1, \Phi_1, V_1)/r\Phi_1$ is that Φ_1 be negative; in particular, E_1 may have a nonzero top. To classify tops of embeddings in definite spaces is not easier than to classify tops of arbitrary embeddings (embeddings in definite spaces we considered in [2]). But if $n^-(E, \Phi) < \infty$ the situation changes drastically:

Corollary 3. If $n := n^-(E, \Phi) < \infty$, $n_0 = n^+(E, \Phi)$ there are $\frac{1}{6}(n+1)(n+2)(n+3)$ orbits of maximal positive definite (maximal positive) subspaces of (E, Φ) , one of them $(n+1)$ consisting of 1-closed subspaces, and $n+1$ orbits consisting of 1-dense subspaces.

Proof. By Corollary 1 we have $n = \dim E/r\phi_1$. Hence the only infinite cardinal number in $V_1(E)$ (Figure 7) is b_0 . This entails that $S^\perp = V^{\perp\perp} + V^\perp$ and $V(E) = V_1(E)$, so that $V(E)$ now looks as in Figure 8 :



$V(E)$ when V is maximal
positive definite and $n^-(E) < \infty$

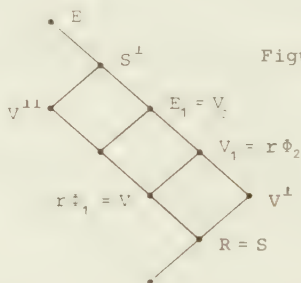


Figure 8

$V(E)$ when V is maximal
positive and $n^-(E) < \infty$

We read off that $n = \dim S + \dim S^\perp/E_1 + \dim E_1/V_1 + \dim V^\perp/S$. By IV.4 Theorem 2 we may partition n into these four summands in an arbitrary way whereas $(V_1, \phi_1)/R_1$ and $(V_2, \phi_2)/R_2$ must be chosen negative definite. Therefore the number of orbits is $\frac{1}{6}(n+1)(n+2)(n+3)$. The remaining assertions follow in the same manner.

Example. If $n^-(E, \phi) = 1$ there is exactly one orbit of maximal positive definite subspaces V which are neither $\perp$ -closed nor $\perp$ -dense. Concretely we can obtain V as follows. Let $E = (Ks \oplus Kt) \oplus \bigoplus_{i \in \mathbb{N}} K e_i$ where $(e_i)_{i \in \mathbb{N}}$ is an orthonormal system and $\phi(s, s) = 0 = \phi(t, t)$, $\phi(s, t) = 1$. Put $V := \bigoplus_{i \in \mathbb{N}} K(e_i + s)$. One has to verify that $V^\perp = Ks$ and $E_1 = V \oplus Ks = r\phi_1$.

3. Hyperbolic subspaces

If both of V/R and $V^\perp/S$ are hyperbolic (if infinite dimensional) then we are in the situation with "sufficiently many" isotropic vectors first solved by Gross in [9] (for arbitrary base fields; see also [12])

Chap. V). His result was that such an embedding is determined by the isometry types of V/R and $V^\perp/S$ (if finite dimensional) and the indices in Kaplansky's 14-element lattice (see loc. cit. or [14]). Our Congruence Theorem IV.3 gives the same issue: $V(E)$ obviously boils down to Kaplansky's lattice. It is, however, interesting to see what happens if only one of $V/R, V^\perp/S$ is required to be hyperbolic.

Consider first the case where V/R is hyperbolic. By IV.4 Lemma 3 all but the following indices must vanish:

$$a_0, b_0, b_1, c_0, d_0, f_0, g_0, h_0.$$

The lattice $V(E)$ (Figure 9) contains only one subspace beyond Kaplansky's lattice, to wit, the space $F = C(V^\perp)$; the index $\dim S^\perp/V^{\perp\perp} + V^\perp$ of the isotropic case is split into g_0, h_0 .

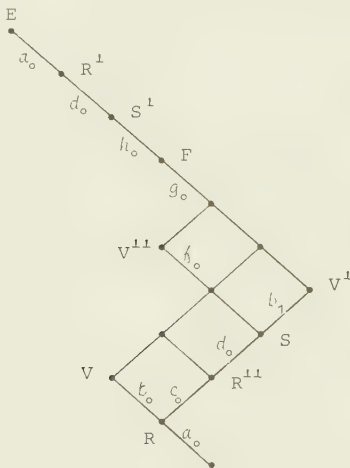


Figure 9 :

$V(E)$ when $(V, \phi)/R$
is hyperbolic

The congruence type of such an embedding is fixed by the above indices and the isometry type of $(V^\perp, \phi)/S$.

The case where $V^\perp/S$ is hyperbolic is somewhat more involved. The indices which may differ from zero are (again by IV.4 Lemma 3)

$$a_0, a_1, b_0, b_1, b_2, c_0, c_1, d_0, d_1, e_0, f_0, f_1, g_1, h_0, h_1, i_0$$

so that $V(E)$ is represented by Figure 10. To fix the congruence type of E one has to specify, apart from these indices, the isometry types of $(V, \phi)/R$ and $(V_1^{\perp}, \phi_1)/S_1$.

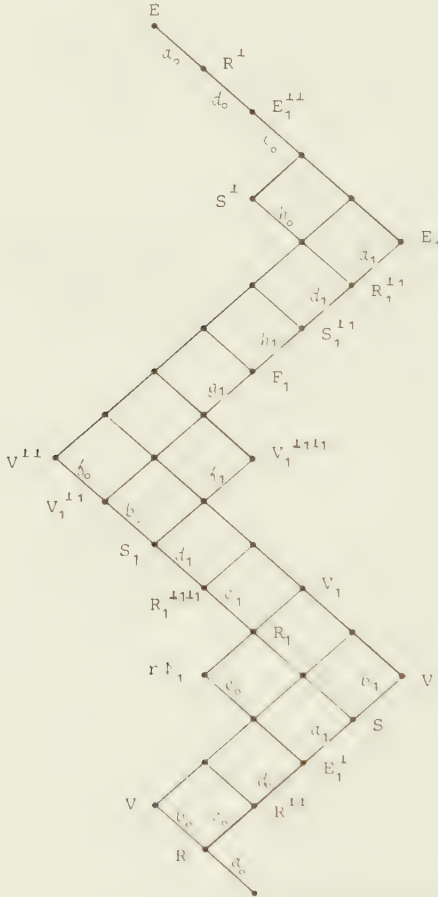


Figure 10 : $V(E)$ when $(V^{\perp}, \phi)/S$ is hyperbolic

4. Subspaces of finite codimension

We fix the space (E, Φ) and consider subspaces $V \subset E$ with $\dim E/V < \infty$. The corresponding lattice $V(E)$ is shown in Figure 11 :

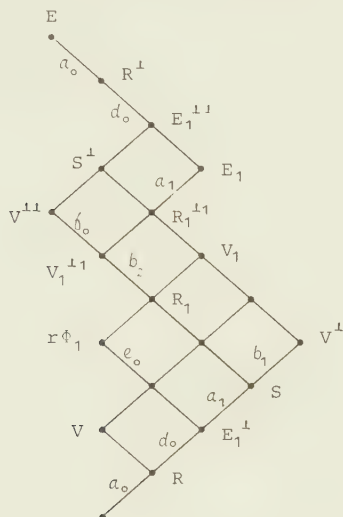


Figure 11 : $V(E)$ when $\dim E/V < \infty$

If (E, Φ) is hyperbolic then so is $(V, \Phi)/R$, a trivial situation covered by the preceding section. Hence assume that $n^+(E) = \aleph_0 > n^-(E)$. For $i = 0, 1, 2$ put $n_i^- := n^-(V_i, \Phi_i)$, $n_i^+ := n^+(V_i, \Phi_i)$. Then the orbit of V under the orthogonal group of (E, Φ) is completely characterized by the natural numbers

$$a_0, a_1, n_0^-, n_1^-, n_1^+, n_2^-, n_2^+, d_0, e_0, \delta_0$$

where

$$\dim E/V = a_0 + 2a_1 + n_1^- + n_1^+ + n_2^- + n_2^+ + 2d_0 + e_0 + \delta_0$$

$$n^-(E, \Phi) = a_0 + a_1 + n_0^- + n_1^- + n_2^- + d_0 + \delta_0.$$

For the second of these equations cf. I.2 (3).

Corollary. There are only finitely many orbits of subspaces of a fixed finite codimension in E .

CHAPTER VI

EXTENDING ISOMETRIES

The concept of ghost form introduced in I.2 has been the key to our treatment of the congruence problem in the preceding chapters. Now we are going to show that ghost forms can also be used to provide criteria deciding whether a given isometry $\varphi: V \xrightarrow{\sim} V'$ between subspaces $V \subset (E, \phi)$, $V' \subset (E', \phi')$ can be extended to an isometry $E \xrightarrow{\sim} E'$. As an application, we shall add a digression on orthogonal separation of pairs of subspaces. As in the preceding chapters, spaces are over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

1. The Extension Theorem

The reader is assumed to be familiar with Chapt. X of [12] where the extension problem is solved for those cases where $V^\perp/S$ is hyperbolic or finite dimensional (the first result of this kind was obtained by Allenspach [1]). In particular, we shall use the elementary properties of dual pairs and linear topologies without further reference (see loc. cit. Sections 1,2 or [15], §§ 10-13). If A, B are subspaces of (E, ϕ) we denote by $\sigma(A, \phi, B)$ or $\sigma(A, B)$ the weak linear topology on A induced by the pairing $\phi|_{A \times B}: A \times B \rightarrow K$ (which may be degenerate). Thus, the filter of zero-neighbourhoods in the topology $\sigma(A, B)$ has as a basis the set $\{F^\perp \cap A \mid F \subset B, \dim F < \infty\}$. Now let

$$\varphi: V \xrightarrow{\sim} V'$$

be an isometry between subspaces $V \subset (E, \phi)$, $V' \subset (E', \phi')$. A necessary condition for φ to extend to an isometry $E \xrightarrow{\sim} E'$ is that φ be a homeomorphism with respect to the topologies $\sigma(V, E) = \sigma(E, E)|_V$ on V and $\sigma(V', E')$ on V' ; we shall render this briefly by saying that φ is weakly homeomorphic. Whether or not φ extends to a weakly homeomor-

phic isometry $V^{\perp\perp} \xrightarrow{\sim} V'^{\perp\perp}$ is a purely topological question: If there is an extension $\varphi_0: V^{\perp\perp} \rightarrow V'^{\perp\perp}$ which is a weak homeomorphism then φ_0 is automatically an isometry. This follows since $V^{\perp\perp}$ is the closure of V in the topology $\sigma(E, E)$, and ϕ is separately continuous (cf. also the beginning of X.6 in [12]). Therefore we shall generally assume that V and V' are $\perp$ -closed (= weakly closed). To say that φ is weakly homeomorphic is tantamount to saying that φ admits a transpose isomorphism

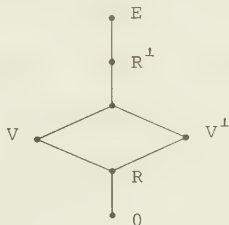
$$\begin{aligned} \varphi^t: E'/V'^{\perp} &\longrightarrow E/V^{\perp} \\ \phi'(\varphi v, x') &= \phi(v, \varphi^t(x' + V'^{\perp})) \quad \text{for } v \in V, x' \in E'. \end{aligned}$$

Finally, recall the notations $F := C(E, \phi, V^{\perp})$, $\Psi := \phi_{V^{\perp}}$ which may be interpreted as a form on $F/V^{\perp}$.

Theorem 1. Let $\varphi: V \rightarrow V'$ be an isometry between the $\perp$ -closed subspaces $V \subset (E, \phi)$, $V' \subset (E', \phi')$. In order that φ extend to an isometry $E \xrightarrow{\sim} E'$ the following are necessary and sufficient:

- (i) $(V^{\perp}, \phi)/R \cong (V'^{\perp}, \phi')/R'$,
 - (ii) φ is weakly homeomorphic,
 - (iii) the restriction of φ^t to $F'/V'^{\perp}$ is an isometry
- $$(F'/V'^{\perp}, \Psi') \xrightarrow{\sim} (F/V^{\perp}, \Psi).$$

Proof. The necessity of the conditions being obvious, let us prove their sufficiency. We shall use the lattice method with respect to Kaplansky's lattice which here is reduced to



Observe that $(\varphi^t)^{-1}$ maps R^1/V^1 and $V + V^1/V^1$ onto the corresponding subspaces of E'/V'^1 so that the indices in the above lattice agree with the corresponding indices in the lattice of V' . In the notations of I.3.3 let G be the collection of all $(T: X \rightarrow X') \in F$ which satisfy

- (1) $X \cap V \rightarrow X' \cap V'$ coincides with the restriction of φ ,
- (2) $X/X \cap V^1 \rightarrow X'/X' \cap V'^1$ coincides with the restriction of $(\varphi^t)^{-1}$.

As to the proof of the Ping-Pong property, we may assume that $y \in D \setminus (X + D_0)$ where D is $+-$ irreducible and D_0 its immediate predecessor. Let us inspect the individual cases.

$D \not\supseteq V^1$: Choose a preliminary y' with $\varphi^t(y' + V^1) = y + V^1$. Then $y' \in D' \setminus (X' + D'_0)$ and y' has the correct angles on $X' \cap V'$. Hence the remaining angles can be adjusted modulo V'^1 (since $(V'^1)^1 = V'$). We are also allowed to change y modulo V^1 , so that by (i) it is easy to arrange for y and y' to have the same length. (If $\dim V^1/R < \infty$ then $D = E \neq R^1$ and we may proceed as in IV.3.15).

$D \subset V$: Choose $y' := \varphi y$. There is no other choice if we want to save (1) for the extension $T_1: X \oplus Ky \rightarrow X' \oplus Ky'$ of T . By (2) the vector y' has the correct angles on all of X' and, of course, it has the correct length. Finally, since φ is an isometry it induces the same map $V + V^1/V^1 \rightarrow V' + V'^1/V'^1$ as $(\varphi^t)^{-1}$, so that (2) is also inherited by T_1 .

$D = V^1$: This case can be reduced to Lemma 1 in II.1. Indeed, we may apply the same arguments as in IV.3.9 since, by (iii) and (2), $T(X \cap F) = X' \cap F'$ and $T: X \cap F \rightarrow X' \cap F'$ respects the form Ψ .

The proof of Theorem 1 is thereby complete. The reader may have noticed that our method differs from that employed in [1] and [12], X.5. There it was verified that finite extensions $\varphi_1: V_1 \rightarrow V'_1$ of φ (i.e., $\dim V_1/V < \infty$) again satisfy (i) and (ii) in Theorem 1;

though feasible it would here be troublesome to show that condition (iii) is inherited by φ_1 . However, it is clear that our method also yields the results in [12], X.5.

For the sake of illustration we add a few observations concerning isometries between not necessarily closed subspaces in special situations.

Remarks. 1. If V and V' are not assumed to be $\perp$ -closed one has to add in Theorem 1 the condition

$$(3) \quad \varphi^t \text{ is a homeomorphism for the quotient topologies of } \sigma(E', V'^{\perp\perp}) \text{ on } E'/V'^{\perp} \text{ and of } \sigma(E, V^{\perp\perp}) \text{ on } E/V^{\perp}$$

(the quotient topology of $\sigma(E, V^{\perp\perp})$ coincides with the quotient topology of $\sigma(E, E)$). For, this means exactly that φ^t admits a transpose $\varphi_0 : V^{\perp\perp} \xrightarrow{\sim} V'^{\perp\perp}$ which is weakly homeomorphic; furthermore φ_0 extends φ and $\varphi_0^t = \varphi^t$. By the remarks preceding the theorem φ_0 is isometric; hence by (3) we are reduced to the $\perp$ -closed case. If $F = E$ and $F' = E'$ condition (3) may be replaced by the more natural (and obviously necessary) requirement

$$(4) \quad \varphi \text{ is a homeomorphism for } \sigma(V, V^{\perp\perp}), \sigma(V', V'^{\perp\perp}).$$

Indeed, if (4) holds then φ^t maps $V'^{\perp\perp} + V'^{\perp}/V'^{\perp}$ onto $V^{\perp\perp} + V^{\perp}/V^{\perp}$ so that, by (iii), φ^t transforms the topology $\sigma(F'/V'^{\perp}, \Psi', V'^{\perp\perp} + V'^{\perp}/V'^{\perp})$ into the corresponding topology on $F/V^{\perp}$. But from I.2.(1) we infer that $\sigma(F/V^{\perp}, \Psi, V^{\perp\perp} + V^{\perp}/V^{\perp})$ is the quotient topology of $\sigma(F, \Phi, V^{\perp\perp})$. Hence if $F = E$, $F' = E'$ then (4) implies (3). This remark applies whenever (E, Φ) , (E', Φ') are definite.

2. Assume that $F = E_1$, $F' = E'_1$. Then φ^t respects Ψ if and only if it respects Φ_2 . For, we have $E_2 = F \cap E_1 = F$ and, for $x, y \in F$,

$\phi_2(x, y) = \phi_1(x, y) - (\phi_1 V_1 \phi_1)(x, y) = \phi(x, y) - (\phi V \phi)(x, y) - (\phi V^\perp \phi)(x, y) =$
 $= \psi(x, y) - (\phi V \phi)(x, y)$, and φ^t respects the form $\phi V \phi$ on $E_1/V^\perp$ automatically. To have a specific example, let (E, ϕ) and (E', ϕ') be positive definite. Then $F = E_1 = E$ and $\phi_2 = \phi_{V+V^\perp}$. Hence if the dense subspace $V + V^\perp \subset E$ is parseval, i.e. $r\phi_2 = E$, cf. IV.4.2 (in which case we also say that V is parseval) then the metric requirement (iii) in Theorem 1 disappears. We have proved

Corollary (Schneider [21]; cf. [12], XII.16.8). Let $(E, \phi), (E', \phi')$ be positive definite and consider an isometry $\varphi: V \xrightarrow{\sim} V'$ between parseval subspaces $V \subset E$, $V' \subset E'$. Then φ extends to an isometry $E \xrightarrow{\sim} E'$ if and only if $\dim V^\perp = \dim V'^\perp$ and φ is homeomorphic for the two topologies $\sigma(V, E)$, $\sigma(V, V^{\perp\perp})$ on V , $\sigma(V', E')$, $\sigma(V', V'^{\perp\perp})$ on V' .

2. Orthogonal separation

If A, B is an orthogonal pair of subspaces $(A \perp B)$ of (E, ϕ) the question whether A and B are orthogonally separated in E (i.e. $E = G \overset{+}{\oplus} H$ with $A \subset G$, $B \subset H$) suggests itself. The question is also interesting with regard to the congruence problem of orthogonal pairs of subspaces (Corollary 1 below). The following two conditions are obviously necessary for A, B to be orthogonally separated:

$$(5) \quad A^\perp + B^\perp = E$$

$$(6) \quad (A+B)^{\perp\perp} = A^{\perp\perp} + B^{\perp\perp}.$$

Conversely, (5) and (6) suffice to separate A and B whenever $(A+B)^\perp / (A+B)^\perp \cap (A+B)^{\perp\perp}$ is finite dimensional or hyperbolic (this holds in N_0 -dimensional trace-valued spaces over arbitrary base fields, see [12], X.7). Our next result gives a precise criterion for the spaces considered here.

Theorem 2. The subspaces $A \perp B$ of (E, Φ) are orthogonally separated in E if and only if (5), (6) and the following condition (7) hold (we put $V := A+B$, $F := C(V)^\perp$, $\Psi := \Phi|_{V^\perp}$).

$$(7) \quad F = F \cap A^\perp + F \cap B^\perp \quad \text{and} \quad F \cap A^\perp \perp_\Psi F \cap B^\perp.$$

Proof. If A, B are separated then (7) is obtained by a straightforward verification, so that we may turn to the sufficiency of the conditions. We might assume that A, B are $\perp$ -closed because (5), (6) and (7) retain their validity if we replace A and B by $A^{\perp\perp}$ and $B^{\perp\perp}$; still, let us dwell on the general situation for the benefit of Corollary 1 below. Set $S_A := A^\perp \cap A^{\perp\perp}$, $S_B := B^\perp \cap B^{\perp\perp}$; hence $S = V^\perp \cap V^{\perp\perp} = S_A \oplus S_B$ by (5) and (6). Now define

$$(E', \Phi') := (B^\perp, \Phi)/S_B \overset{1}{\oplus} (A^\perp, \Phi)/S_A \quad (\text{external sum})$$

$$A' := A + S_B/S_B$$

$$B' := B + S_A/S_A.$$

We have $A'^{\perp\perp} = A^{\perp\perp} + S_B/S_B$, $B'^{\perp\perp} = B^{\perp\perp} + S_A/S_A$, and since $A^{\perp\perp} \cap S_B = B^{\perp\perp} \cap S_A = 0$ by (5) the canonical map

$$\varphi : V^{\perp\perp} = A^{\perp\perp} \oplus B^{\perp\perp} \xrightarrow{\sim} A'^{\perp\perp} \oplus B'^{\perp\perp} = V'^{\perp\perp}$$

is an isometry which maps V onto V' . The theorem will follow if we succeed in extending φ to an isometry $E \xrightarrow{\sim} E'$. In other words, we have to check the conditions (i)-(iii) in Theorem 1. As to (i), we compute that $V'^\perp = A^\perp \cap B^\perp/S_B \overset{1}{\oplus} B^\perp \cap A^\perp/S_A = V^\perp/S_B \overset{1}{\oplus} V^\perp/S_A$ and $S' = V'^\perp \cap V'^{\perp\perp} = S_A + S_B/S_B \overset{1}{\oplus} S_B + S_A/S_A = S/S_B \overset{1}{\oplus} S/S_A$; hence $V'^\perp/S' \cong V^\perp/S \overset{1}{\oplus} V^\perp/S$ which need not be isometric to $V^\perp/S$. To remedy this, let us jump back to the beginning of the proof: Choose a fundamental decomposition of a supplement W of S in $V^\perp$, $W = W_- \overset{1}{\oplus} W_+$, and let M be the sum of those summands which are finite dimensional. Then $A, B \subset M^\perp$ and (5), (6), (7)

remain true if we replace E by its orthogonal summand $M^\perp$. This manipulation takes care of (i). Instead of verifying condition (ii) we prefer to directly determine the transpose $\varphi^\perp$ of φ : It is the canonical map

$$E'/V'^\perp = B^\perp/V^\perp \oplus A^\perp/V^\perp \longrightarrow E/V^\perp$$

which is an isomorphism by (5) and because $A^\perp \cap B^\perp = V^\perp$. There remains condition (iii). First observe that $F' = C(V'^\perp) = F \cap B^\perp/S_B \oplus F \cap A^\perp/S_A$ (since $V'^\perp = V^\perp/S_B \oplus V^\perp/S_A$) so that $\varphi^\perp$ maps $F'/V'^\perp$ onto $(F \cap B^\perp + F \cap A^\perp)/V^\perp = F/V^\perp$, by (7). Furthermore, $(F'/V'^\perp, \Psi') = (F \cap B^\perp/V^\perp, \Psi) \oplus (F \cap A^\perp/V^\perp, \Psi)$, hence (7) implies that $\varphi^\perp$ is $\Psi' - \Psi$ -isometric.

The proof of Theorem 2 actually gives us more precise information; it shows that the splittings of E which separate A and B may be chosen in a very particular way:

$$E = G \oplus H \oplus M \quad \text{with} \quad A \subset G, B \subset H, \dim M < \infty$$

such that we have the following congruences of embeddings:

$$(8) \quad \begin{aligned} (G \oplus M, \Phi, A) &\approx (B^\perp, \dot{\Phi}, A + S_B)/S_B \\ (H \oplus M, \Phi, B) &\approx (A^\perp, \dot{\Phi}, B + S_A)/S_A \end{aligned}$$

and where the isometry type of M may be specified to depend only on the isometry type of $V^\perp/S$. But $V^\perp/S$ occurs also in the embeddings on the right hand side in (8) (as $\dot{V}^\perp/\dot{S}$ if these embeddings are abbreviated by $(\dot{E}, \dot{\Phi}, \dot{V})$).

The issue of this reasoning is

Corollary 1. Let $A \perp B$, $A' \perp B'$ be two orthogonal pairs of subspaces of (E, Φ) that are separated in E (i.e., both pairs satisfy (5), (6) and (7)). Then the two pairs are congruent in E (i.e., there is an isometry $T: E \xrightarrow{\sim} E$ such that $TA = A'$, $TB = B'$) if and only if the

two embeddings on the right hand side of (8) are congruent to the corresponding embeddings attached to the pair A', B' .

This is almost clear by what we have said before; we only have to take into account that a congruence $(G \oplus^{\perp} M, \phi, A) \cong (G' \oplus^{\perp} M', \phi, A')$ entails a congruence $(G, \phi, A) \cong (G', \phi, A')$ since $M \subset A^{\perp}$, $M' \subset A'^{\perp}$ and $M \cong M'$, $\dim M < \infty$.

As an illustration to the topics of this section we have reproduced the $\perp$ -stable lattice $V(A, B)$ generated by an orthogonal pair with (5) and (6) (Figure 12); it was computed by Brand [7] (cf. [12], VI.2). The Kaplansky lattices of the two embeddings on the right hand side of (8) correspond to the intervals $[S_B, B^{\perp}]$, $[S_A, A^{\perp}]$ of $V(A, B)$, and it is obvious from the diagram that the indices in these two intervals determine the indices in the whole lattice.

Corollary 2. Each orthogonal pair (A, B) with (5) and (6) and A totally isotropic is orthogonally separated in E .

Proof. We claim that condition (7) in Theorem 2 is automatically satisfied. From $A \subset A^{\perp}$ we get that $F \subset A^{\perp}$ whence the first half of (7) is plain. Furthermore $F \cap B^{\perp} \subset A^{\perp} \cap B^{\perp} = V^{\perp} = r\psi$ which implies the second half of (7).

We hasten to add that the assertion of Corollary 2 holds in arbitrary $\aleph_0$ -dimensional spaces (for trace-valued spaces this was proved in [3]; the general case can be found in [17]).

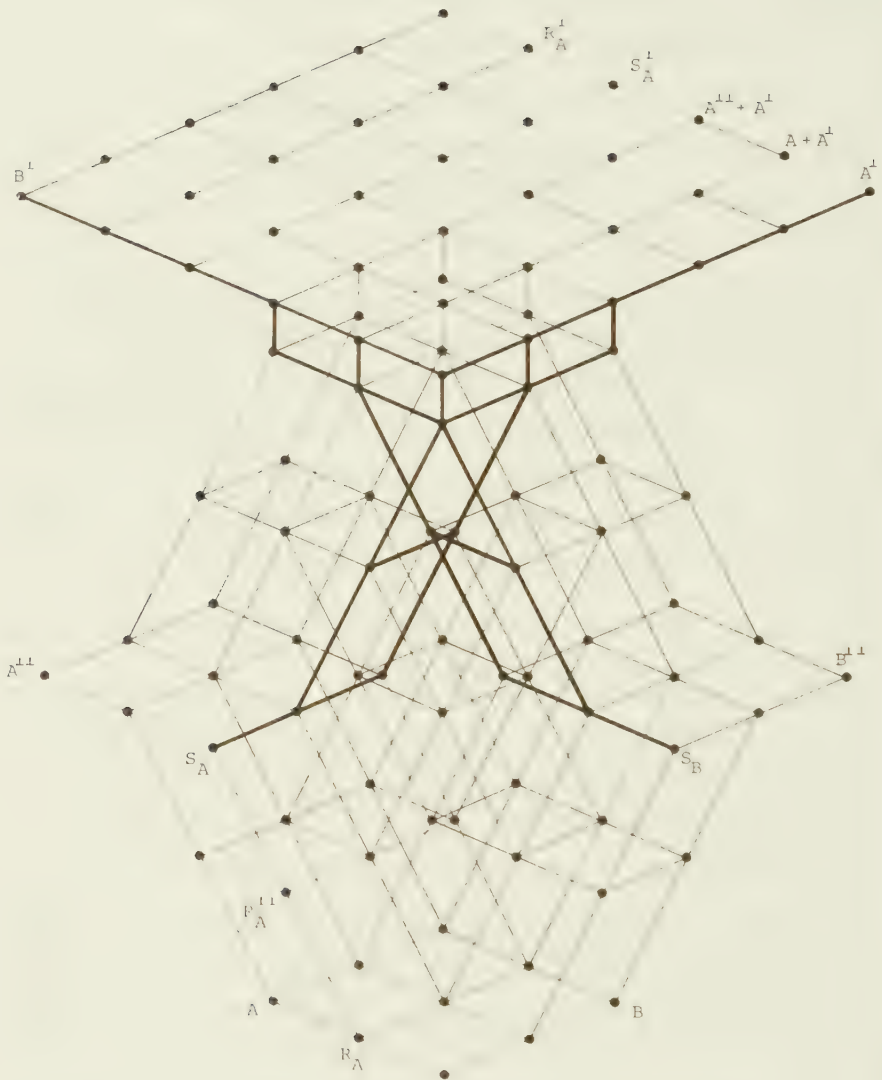


Figure 12 : 1-stable lattice generated by a pair A, B with
 $A \perp B$, $A^{\perp} + B^{\perp} = E$, $(A+B)^{\perp\perp} = A^{\perp\perp} + B^{\perp\perp}$

3. Remark on symplectic separation

A pair $A \subset A^\perp$, $B \subset B^\perp$ of totally isotropic subspaces of (E, Φ) is said to be symplectically separated in E if there is a Witt decomposition $E = (G \oplus H) \oplus^\perp M$ (i.e., $G \subset G^\perp$, $H \subset H^\perp$) with $A \subset G$, $B \subset H$. In many aspects the theory of symplectic separation runs parallel to that of orthogonal separation; however, unlike the problem of orthogonal separation, the problem of symplectic separation admits a uniform solution irrespective of the nature of the base field. This is what we want to point out here.

Theorem 3. Let (E, Φ) be a nondegenerate trace-valued space of dimension N_0 over an arbitrary base field, and let $A \subset A^\perp$, $B \subset B^\perp$ be a pair of totally isotropic subspaces of E . In order that A, B be symplectically separated in E it is necessary and sufficient that A, B satisfy (5), (6) and

$$(9) \quad (A+B)^\perp / (A+B)^\perp \cap (A+B)^{\perp\perp} \text{ is hyperbolic or } (A+B)^{\perp\perp} + (A+B)^\perp \text{ is } \perp\text{-closed.}$$

Proof. We may assume that A, B are $\perp$ -closed, and we put again $V := A+B$. To prove the necessity of (9) (that of (5), (6) is obvious) consider a symplectic separation $E = (G \oplus H) \oplus^\perp M$ of A, B . We find $V^\perp = B^\perp \cap G \oplus A^\perp \cap H \oplus M$, $S = B^\perp \cap A \oplus A^\perp \cap B$, whence $V^\perp / S \cong (B^\perp \cap G / B^\perp \cap A \oplus A^\perp \cap H / A^\perp \cap B) \oplus^\perp M$. Hence there is a supplement P of S in $V^\perp$ of the form

$$P = (\text{orthogonal summand in } E) \oplus^\perp (\text{hyperbolic space})$$

so that P itself is either hyperbolic or an orthogonal summand in E . In the latter case we have $V + V^\perp = S^\perp$ which proves (9). Conversely, assume that (5), (6) and (9) hold and consider first the case where $V + V^\perp$ is $\perp$ -closed. A Witt decomposition with respect to S

($E = (S \oplus S')^{\perp} \oplus E_0$, $S' \subset S'^{\perp}$) reveals that there is a supplement P of S in $V^{\perp}$ (to wit, $P = V^{\perp} \cap E_0$) which is an orthogonal summand in E . Hence we see that we may reduce our problem to the case where $\dim V^{\perp}/S \in \{0, \infty\}$ and $V^{\perp}/S$ is hyperbolic, by splitting off a suitable supplement of S in $V^{\perp}$, if necessary. Now it follows from Theorem 4 in [12], VI.4 that A, B are symplectically separated in E . Still, we remark that it is also possible to argue along the lines of the proof of Theorem 2: Set

$$E' := A^{\perp}/A^{\perp} \cap B \oplus B^{\perp}/B^{\perp} \cap A$$

$$\phi'(x+y, x'+y') := \phi(x, y') + \phi(y, x') \quad (x, x' \in A^{\perp}, y, y' \in B^{\perp})$$

$$A' := A + A^{\perp} \cap B/A^{\perp} \cap B$$

$$B' := B + B^{\perp} \cap A/B^{\perp} \cap A.$$

Then the canonical map $\varphi: A \oplus B \rightarrow A' \oplus B'$ is an isometry and can be shown to extend to an isometry $E \xrightarrow{\sim} E'$ on grounds of the analogue to Theorem 1 in "isotropic" situations. Indeed, $V^{\perp}/S' = V^{\perp}/S \oplus V^{\perp}/S$ (the summands being totally isotropic under ϕ') is isometric to $(V^{\perp}, \phi)/S$ by our assumption on $V^{\perp}/S$, and the canonical isomorphism $E'/V^{\perp} = A^{\perp}/V^{\perp} \oplus B^{\perp}/V^{\perp} \rightarrow E/V^{\perp}$ is the transpose of φ whence φ is weakly homeomorphic.

CHAPTER VII

EMBEDDINGS OVER SUBFIELDS OF THE REALS

1. Introduction

Some results of previous chapters admit natural generalizations to the case of orderable base fields other than $\mathbb{R}$. Although these congruence theorems do not lead to a classification of a reasonably extensive class of embeddings (in fact, "unclassifiable" invariants in the sense of the Appendix appear already in the case of $\mathbb{A}$ -dense subspaces) they deserve interest for at least two reasons: Firstly, they locate the obstacles forbidding a classification and, secondly, they emphasize the singular position (with regard to the congruence problem) of $\mathbb{R}$ among other orderable base fields, a position which derives from the fact that $\mathbb{R}$ is the only orderable field in which every bounded increasing sequence converges. The results we are going to state bear an exemplary character; we have striven neither for completeness nor for utmost generality. For instance, we shall stick to symmetric forms over commutative orderable base fields rather than to hermitean forms over quadratic and quaternionic extensions of such fields. The proofs have been reduced to hints that should enable the reader who has read the earlier chapters to carry out the arguments in detail.

Our assumptions on the commutative orderable base field k are the following:

- (1) k has only finitely many orderings and each of these is archimedean,
- (2) each $\aleph_0$ -dimensional symmetric nondegenerate k -space (E, ϕ) with $\|E\| \subset \mathbb{Z} k^2$ represents 1 (and hence admits an orthonormal basis).

Here $\|E\| = \{\phi(x, x) \mid x \in E \setminus \{0\}\}$ and Σk^2 is the set of sums of squares in k . Examples of fields that satisfy (1) and (2) are furnished by the (formally real) algebraic number fields ((2) follows from Hasse theory) and by the real closed subfields of $\mathbb{R}$. It is well known that the archimedean orderings P of a field k correspond bijectively to the homomorphisms $\eta: k \rightarrow \mathbb{R}$ by $\eta \mapsto P_\eta = \{\alpha \in k \mid \eta\alpha > 0\}$, orderings being identified with their cones of strictly positive elements. Let $X(k)$ be the set of all homomorphisms $k \rightarrow \mathbb{R}$; then $\Sigma k^2 = \bigcap_{\eta \in X(k)} P_\eta \cup \{0\}$ by (1) and Artin-Schreier theory.

Before plunging into the congruence problem of embeddings over fields k with (1) and (2) it is necessary to shortly recall the classification of spaces over such a field (see [16], [4] or [12] Chap. XI). Two (symmetric nondegenerate N_0 -dimensional) spaces $(E, \phi), (E', \phi')$ are isometric if and only if for each $\eta \in X(k)$ the $\mathbb{R}$ -ifications of ϕ and ϕ' via η are isometric; furthermore each space (E, ϕ) is quasistable: (E, ϕ) is termed stable if $(E, \phi) \cong N_0(E, \phi) = (E, \phi) \oplus (E, \phi) \oplus \dots$ and quasistable if it contains a stable orthogonal summand of finite codimension. A stable space (E, ϕ) is uniquely determined by the set $\|E\|$ which has the shape $\|E\| = \alpha \cdot \bigcap_{\eta \in I} P_\eta$ for some subset $I \subset X(k)$ and each $\alpha \in \|E\| \setminus \{0\}$. If we fix a family $(\epsilon_\eta)_{\eta \in X(k)}$ of elements of k such that $\mu(\epsilon_\eta) < 0$ if and only if $\mu = \eta$ (which exists by the Artin-Whaples approximation theorem) we may choose $\alpha = \prod_{\eta \in J} \epsilon_\eta$ for some subset $J \subset I$. From all this we see that an arbitrary space (E, ϕ) takes on the form

$$(3) \quad (E, \phi) \cong \prod_{\eta \in J} \epsilon_\eta \cdot \left(\bigoplus_{\eta \in I} n_\eta \langle \epsilon_\eta \rangle \oplus (E_0, \phi_0) \right)$$

where $J \subset I \subset X(k)$, $n_\eta \in \mathbb{N}$ for $\eta \in I$, (E_0, ϕ_0) stable with $\|E_0\| = \bigcap_{\eta \in I} P_\eta$. (By $n\langle \alpha \rangle$ we denote a space of dimension n with an orthogonal basis (e_i) such that $\phi(e_i, e_i) = \alpha$.)

Finally we wish to compare our assumptions (1) and (2) with those made by Gross in Chap. XII of [12]. Gross considers an ordered field $(k, <)$ together with a strongly universal k -space (E, ϕ) with orthonormal basis (p. 269). Strong universality of (E, ϕ) is tantamount to (2). Gross assumes further that each positive element of k is a sum of at most m squares, m a fixed natural number ((4) on p. 278). This means that k admits only one ordering and has finite Pythagoras number $^*)$. In contrast to (1) this ordering is not assumed to be archimedean; however, this has to be paid for inasmuch as the results must be restricted to embeddings admitting a "standard basis" (p. 272).

2. An extension lemma

We consider a μ -dense embedding $V \subset (E, \phi)$ of finite codimension, $\dim E/V = m < \infty$. Because splitting off finite dimensional orthogonal summands of V and scaling the forms are harmless manipulations (with regard to what we have in mind) we shall assume that V is stable and $1 \in \|V\|$, hence $\|V\| = \bigcap_{\mu \in I} P_\mu$ for some $I \subset X(k)$ (see (3)). Unfortunately, the extension lemma in II.1 has no analogue in the present situation. The reason for this can be found in the fact that the space $C(V)$, although meaningful, loses its importance. Indeed, for each $\mu \in I$ we may define $C_\mu(V) := E \cap C(\mathbb{R} \otimes_\mu V) = \{x \in E \mid \text{the sum } \sum_{i \in \mathbb{N}} \mu(\phi(x, v_i)^2 \phi(v_i, v_i)^{-1}) \text{ converges in } \mathbb{R}\}$ where $(v_i)_{i \in \mathbb{N}}$ is any orthogonal basis of V ; but $C_\mu(V)$ is not compatible with extending the scalars via μ . Let us give an example for this phenomenon: View μ as an inclusion $k \subset \mathbb{R}$ and pick $\alpha \in \mathbb{R} \setminus k$, let V have an orthonormal basis $(v_i)_{i \in \mathbb{N}}$ and put $E := V \oplus kx \oplus ky$. Define $\phi(x, v_i) := -1$ for all i and let $\phi(y, v_i) \in k$ be such that

$^*)$ The existence of uniquely ordered subfields of $\mathbb{R}$ with infinite Pythagoras number has been proved by Prestel [18].

$\sum_{i \in \mathbb{N}} (\phi(y, v_i) - \alpha)^2 < \infty$. Then, $\sum_{i \in \mathbb{N}} \phi(\xi x + y, v_i)^2 < \infty$ is equivalent to $\xi = \alpha$ whence $C_\mu(V) = V$ whereas $C(\mathbb{R} \otimes_\mu V) \supsetneq \mathbb{R} \otimes_\mu V$.

Instead of $C(V)$ and ϕ_V we may use the B-matrices introduced in II.2 to obtain an analogue of Proposition 1 in II.2. Fix a basis $x_1, \dots, x_m$ of a supplement X of V in E and let $(v_i)_{i \in \mathbb{N}}$ be an orthogonal basis of V . Define $m \times m$ -matrices A_n, B_n over k by

$$\begin{aligned} A_{n1\kappa} &:= \sum_{i=1}^n \phi(x_i, v_i) \phi(v_i, v_i)^{-1} \phi(v_i, x_\kappa) \quad , \quad 1 \leq \kappa \leq m \\ B_n &:= A_n^{-1} \quad (\text{for large } n, A_n \text{ is invertible}). \end{aligned}$$

For $\mu \in I$ we have $\mu(\phi(v_i, v_i)) > 0$, hence the arguments in II.2.1 show that the sequence of $\mathbb{R}$ -matrices $\mu(B_n)$ converges,

$$\mu_B := \lim_{n \rightarrow \infty} \mu(B_n) \quad .$$

In fact, μ_B is nothing but the B-matrix of II.2.1 associated to $\mathbb{R} \otimes_\mu V$ with respect to $1 \otimes x_1, \dots, 1 \otimes x_m$, as is seen by replacing the v_i by $\mu(\phi(v_i, v_i))^{-\frac{1}{2}} \otimes v_i$.

Lemma 1. Let $E = V \oplus X$, $E' = V' \oplus X'$, $X = kx_1 \oplus \dots \oplus kx_m$, $T: X \xrightarrow{\sim} X'$ an isometry, $x'_i = Tx_i$. Assume that V, V' are μ -dense and stable with $\|v\| = \|v'\| = \bigcap_{\mu \in I} p_\mu$, $I \subset X(k)$. In order that T extend to an isometry $\bar{T}: E \xrightarrow{\sim} E'$ with $\bar{T}V = V'$ it is necessary and sufficient that $\mu_B = \mu_{B'}$ for each $\mu \in I$.

Proof. One has to prove the analogues of Lemma 2 and Lemma 3 in II.2.2. The formulas get somewhat more complicated since here we must not assume that $\phi(v, v) = 1$. Yet we find that the shortest vector $v'(N)$ in $\mathbb{R} \otimes_\mu V_N$ ($V_N = kv'_1 \oplus \dots \oplus kv'_m$, $\mu \in I$) with $\phi'(v'(N), x'_i) = \phi(v, x_i)$ ($i = 1, \dots, m$) is defined over k ,

$$v'(N) = \sum_{i=1}^N \sum_{\kappa=1}^m \phi(v, x_\kappa) B'_{N1\kappa} \phi'(x'_\kappa, v'_1) \phi'(v'_1, v'_1)^{-1} v'_1 \quad ,$$

and that for large N we have ¹⁾ $\Phi'(v'(N), v'(N)) <_{\mu} \Phi(v, v)$.

Since $X(k)$ is finite this inequality holds simultaneously for all $\mu \in I$ if N is sufficiently large. Hence $\Phi(v, v) - \Phi'(v'(N), v'(N)) \in \|\mathbf{v}'\|$ for large N and by the stability of V' there is $z' \in V' \cap v'(N)^{\perp} \cap X^{\perp}$ with $\Phi'(z', z') = \Phi(v, v) - \Phi'(v'(N), v'(N))$. The vector $v' := v'(N) + z'$ is a suitable match for v . The analogue of II.2 Lemma 3 follows from this very lemma by tensoring.

Remark. If $I = \emptyset$ in the lemma the condition on the B -matrices is empty. This is no surprise because V is hyperbolic in this case.

3. $\perp$ -dense and $\perp$ -closed embeddings of finite type

3.1 Let $E = (E, \Phi, V)$ be a $\perp$ -dense embedding and let $I \subset X(k)$ belong to the stable part V_0 of V (see (3)), i.e. $\|\mathbf{V}_0\| = \alpha \cdot \prod_{\mu \in I} p_{\mu}$ for any $\alpha \in \|\mathbf{V}_0\|$. We say that E is of finite type if for each $\mu \in I$ we have $C_{\mu}(V) = E$. This means that for each $x \in E$ and $\mu \in I$ the sum $\sum_{i \in \mathbb{N}} \mu(\Phi(x, v_i)^2 \Phi(v_i, v_i)^{-1})$ converges in $\mathbb{R}$, where $(v_i)_{i \in \mathbb{N}}$ is an arbitrary orthogonal basis of V (we observe that for $\mu \in I$ all but finitely many of the $\mu(\Phi(v_i, v_i))$ have the same sign). Hence we obtain a symmetric k -bilinear map

$$\begin{aligned} \Phi_V : E \times E &\longrightarrow \mathbb{R}^I \\ \Phi_V(x, y)_{\mu} &:= \mu(\Phi(x, y)) - \sum_{i \in \mathbb{N}} \mu(\Phi(x, v_i) \Phi(v_i, v_i)^{-1} \Phi(v_i, y)) \\ &\quad (\mu \in I) \end{aligned}$$

where k operates on $\mathbb{R}^I$ via the canonical map $k \longrightarrow \mathbb{R}^I$. As in the case $k = \mathbb{R}$, Φ_V vanishes identically on $V \times E$. The induced map $E/V \times E/V \longrightarrow \mathbb{R}^I$ is also denoted by Φ_V ; it may be degenerate. We say that $(E/V, \Phi_V)$ and $(E'/V', \Phi'_{V'})$ are isomorphic if there exists a

¹⁾ We write $\alpha <_{\mu} \beta$ for $\mu(\alpha) < \mu(\beta)$

k -isomorphism $\bar{T} : E/V \longrightarrow E'/V'$ which renders the diagram

$$\begin{array}{ccc}
 E/V \times E/V & & \\
 \downarrow \bar{T}, \bar{T} & \searrow \phi_V & \\
 E'/V' \times E'/V' & \nearrow \phi_{V'} & \mathbb{R}^I
 \end{array}$$

commutative. It is not too difficult to show¹⁾ that if $\Psi : F \times F \longrightarrow \mathbb{R}^I$ (F a k -space of countable dimension, $I \subset X(k)$) is an arbitrary symmetric k -bilinear map there exists a $\perp$ -dense embedding E of finite type such that $(E/V, \phi_V) \approx (F, \Psi)$. Hence the classification problem of such embeddings is equivalent to the problem of classifying the objects (F, Ψ) with respect to isomorphism (a wild problem in the sense of the Appendix if $k \neq \mathbb{R}$), by the following

Theorem 1. The congruence class of a $\perp$ -dense embedding of finite type is characterized by

- (a) the isometry class of (V, ϕ) ,
- (b) the isomorphism class of $(E/V, \phi_V)$.

Proof. Fix an isomorphism $\bar{T} : (E/V, \phi_V) \xrightarrow{\sim} (E'/V', \phi_{V'})$ and let V be the chain $\{0, V, E\}$. We proceed along the lines of the proof of I.4 Theorem 1, assuming w.l.o.g. that V is stable: Write V (and V') in normal form, $V = V_{\text{fin}} \dot{\oplus} V_0$ (V_{fin} the finite dimensional summand and V_0 the stable summand in the decomposition (3) of (V, ϕ)) and split off V_{fin} , $E = V_{\text{fin}} \dot{\oplus} E_0$; the invariant (b) is not affected if we replace (E, ϕ, V) by (E_0, ϕ, V_0) . Scaling both forms ϕ and ϕ' with the same $\alpha \in \|V\| = \|V'\|$ we arrive at a situation with $\|V\| = \|V'\| = \bigcap_{\mu \in I} P_\mu$. Now, in the verification of the Ping-Pong pro-

1) Among other things one has to make systematic use of the weak approximation theorem (Artin-Whaples) for archimedean valuations (cf. [12], Appendix II, p. 299, for a similar construction).

erty, the case $D=V$ is reduced to Lemma 1: the matrix ${}^{\mu}B$ in Lemma 1 coincides with the inverse of the $\mathbb{R}$ -matrix $((\phi - \phi_V)(x_i, x_{\kappa})_{\mu})_{1 \leq i, \kappa \leq m}$. In the case $D=E$ we find a preliminary vector $y' \in D' \setminus (X' + D'_0)$ with $y' + V' = \bar{T}(y + V)$ and the correct angles on X' just as in I.4. We claim that $(\phi(y, y) + \|V\|) \cap (\phi(y', y') + \|V'\|) \neq \emptyset$ which allows to rearrange matters such as to have, in addition, $\phi(y, y) = \phi(y', y')$ (since both y and y' may be altered modulo V and V' respectively). But the claim is clear because $\|V\| - \|V'\| = k$.

Examples. 1. If (E, ϕ) is stable then each $\perp$ -dense subspace V of finite codimension is of finite type. For, V is stable with $\|V\| = \|E\|$, and for each $\mu \in I$ the $\mathbb{R}$ -space $\mathbb{R} \otimes_{\mu} (E, \phi)$ is definite.

2. If (E, ϕ) has an orthonormal basis then each $\perp$ -dense subspace is of finite type. We say that V is topologically dense in E if V is dense in E with respect to each of the topologies induced by the "norms" $\hat{\mu} : E \longrightarrow \mathbb{R}, x \longmapsto \hat{\mu}(x) := \mu(\phi(x, x))^{\frac{1}{2}} \quad (\mu \in X(k))$. By a standard argument V is topologically dense if and only if $\phi_V = 0$ (each $x \in E$ is then approximated by the sums $\sum_{i=1}^n \phi(x, v_i) v_i$). Hence Theorem 1 yields the worth-while

Corollary. Let (E, ϕ) possess an orthonormal basis. Topologically dense subspaces V, V' are congruent in E if and only if $\dim E/V = \dim E/V'$.

3.2 We now turn to $\perp$ -closed embeddings E . Assume that $\dim V/R = \dim V^{\perp}/R = \aleph_0$ and let the subsets I_1, I_2 of $X(k)$ belong to the stable parts of $V/R, V^{\perp}/R$ respectively. We say that E is of finite type if for all $\mu \in I_1$ and $\eta \in I_2$ we have $C_{\mu}(V) = R^{\perp} = C_{\eta}(V^{\perp})$. In this case we may define the k -bilinear symmetric maps

$$\begin{aligned}\phi_V &: R^1/V \times R^1/V \longrightarrow \mathbb{R}^{I_1} \\ \phi_{V^\perp} &: R^1/V^\perp \times R^1/V^\perp \longrightarrow \mathbb{R}^{I_2}\end{aligned}$$

in the same way as in 3.1. Since for $x \in V^\perp$, $y \in R^1$ we have that $\phi_V(x, y) = \phi(x, y) \in k$ (identifying k with its image in $\mathbb{R}^{I_1}$ under the canonical map $k \longrightarrow \mathbb{R}^{I_1}$) ϕ_V induces a symmetric k -bilinear map

$$\begin{aligned}\Gamma_1 &: R^1/V + V^\perp \times R^1/V + V^\perp \longrightarrow \mathbb{R}^{I_1}/k, \\ \Gamma_1(x + V + V^\perp, y + V + V^\perp) &:= \phi_V(x, y) + k.\end{aligned}$$

Likewise $\phi_{V^\perp}$ induces $\Gamma_2: R^1/V + V^\perp \times R^1/V + V^\perp \longrightarrow \mathbb{R}^{I_2}/k$.

Finally, let us define

$$\begin{aligned}\Gamma_3 &: R^1/V + V^\perp \times R^1/V + V^\perp \longrightarrow \mathbb{R}^{I_1 \cap I_2} \\ \Gamma_3(x + V + V^\perp, y + V + V^\perp)_\mu &:= \phi_V(x, y)_\mu + \phi_{V^\perp}(x, y)_\mu - \phi(x, y)_\mu \\ &\text{for } \mu \in I_1 \cap I_2\end{aligned}$$

(to fix ideas: if (E, ϕ) has an orthonormal basis then $\Gamma_3 = \phi_{V+V^\perp}$). The map Γ_3 is not independent of Γ_1 and Γ_2 ; its values are determined by Γ_1 and Γ_2 modulo k , more exactly, modulo the image of k in $\mathbb{R}^{I_1 \cap I_2}$. Our next result shows that a $\perp$ -closed embedding E of finite type is characterized by the isomorphism type of the symmetric bilinear map

$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3): R^1/V + V^\perp \times R^1/V + V^\perp \longrightarrow \mathbb{R}^{I_1}/k \times \mathbb{R}^{I_2}/k \times \mathbb{R}^{I_1 \cap I_2}$$

(plus a few "harmless" invariants), provided that E satisfies

$$(4) \quad I_1 \neq \emptyset \neq I_2 \implies I_1 \cap I_2 \neq \emptyset.$$

For example, (4) holds if (E, ϕ) is not hyperbolic or if $\text{card } X(k) = 1$. In a space (E, ϕ) with orthonormal basis each $\perp$ -closed subspace V is of finite type and satisfies (4). For such embeddings we have

Theorem 2 . In the class of $\perp$ -closed embeddings E of finite type with $\dim V/R = \dim V^\perp/R = \infty$ and with (4) the congruence class of an embedding E is characterized by

- (a) $\dim R$,
- (b) the isometry classes of $(V, \Phi)/R$ and $(V^\perp, \Phi)/R$,
- (c) the isomorphism class of Γ .

Proof . Consider two embeddings E, E' of this kind with equal invariants. Splitting off finite dimensional orthogonal summands of E contained in V or $V^\perp$ does not affect the invariants (a) and (c); hence we may assume that the spaces in (b) are stable. Now we use the lattice method to lift an isomorphism $\bar{T}: (R^\perp/V+V^\perp, \Gamma) \xrightarrow{\sim} (R'^\perp/V'+V^\perp, \Gamma')$ to a congruence $E \xrightarrow{\sim} E'$: Choose $V = \{0, R, V, V^\perp, V+V^\perp, R^\perp, E\}$ and let $G \subset F$ consist of all $(T: X \rightarrow X') \in F$ which respect $\Phi_V, \Phi_{V^\perp}$ and which coincide with $\bar{T}$ on $X \cap R^\perp / X \cap (V+V^\perp)$. In the proof of (PP), the cases $D=V$ and $D=V^\perp$ are easily reduced to Lemma 1 and the cases $D=R, D=E$ present no problems, so that we are left with the case $D=R^\perp$. Choose a preliminary y' with $y' + V + V'^\perp = \bar{T}(y + V + V^\perp)$. Both y and y' may be changed modulo $V + V^\perp$ and $V' + V'^\perp$ respectively. The Φ_V - "angles" of y on $X \cap R^\perp$ and the $\Phi_{V^\perp}$ - "length" of y can be adjusted modulo $V^\perp$ since $\Phi_V(y, x) - \Phi_{V'}'(y', Tx)$ and $\Phi_{V^\perp}(y, y) - \Phi_{V'}'(y', y')$ lie in k (because $\bar{T}$ respects Γ_1) : Put $X \cap R^\perp = X \cap V \oplus U$ and choose $z \in V^\perp$ with $\Phi(z, x) = -\Phi_V(y, x) + \Phi_{V'}'(y', Tx)$ for $x \in U$; then $\Phi_V(y+z, x) = \Phi_V(y, x) + \Phi(z, x) = \Phi_{V'}'(y', Tx)$ for all $x \in X \cap R^\perp$. There is also enough room to correct the $\Phi_{V^\perp}$ -length: We have $(X + k y)^\perp \cap V^\perp = (X + k y)^\perp \cap V^\perp$ and $(\Phi_V(y, y) + \|V^\perp\|) \cap (\Phi_{V'}'(y', y') + \|V'^\perp\|) \neq \emptyset$. The $\Phi_{V^\perp}$ -quantities are adjusted in the same way modulo V which does not affect the Φ_V -quantities. Finally we have to care about the Φ -quantities of y . Assume first that $I_1 \neq \emptyset \neq I_2$, hence $I_1 \cap I_2 \neq \emptyset$ by (4). Choosing any

$\mu \in I_1 \cap I_2$ we find that $\mu(\phi(y, x)) = \phi_V(y, x)_\mu + \phi_{V^\perp}(y, x)_\mu - \Gamma_3(y, x)_\mu = \dots = \mu(\phi'(y', Tx))$ for $x \in X \cap R^\perp$, hence $\phi(y, x) = \phi'(y', Tx)$ and similarly $\phi(y, y) = \phi'(y', y')$. The remaining ϕ -angles may be corrected modulo R . If one of I_1, I_2 is empty there is no problem either: Assume, e.g., that $I_1 \neq \emptyset = I_2$. Then the $\phi_{V^\perp}$ -quantities vanish ($R^{\perp 2} = \{0\}$) and the ϕ -angles of y on $X \cap V^\perp$ coincide with the ϕ_V -angles; hence it is possible to adjust the ϕ -quantities modulo V .

To finish this section let us check what the theorem gives us in the case $k = \mathbb{R}$: Both Γ_1 and Γ_2 vanish, and $\Gamma_3 = \phi_V + \phi_{V^\perp} - \phi = \phi - \phi_V \phi - \phi_{V^\perp} \phi = \phi_2$. Hence a $\perp$ -closed embedding of finite type is characterized by $\dim R$ and the isometry types of $(V, \phi)/R$, $(V^\perp, \phi)/R$, and $(E_2, \phi_2)/V + V^\perp$, in accordance with V.1 (there the last invariant is split into $(E_2, \phi_2)/r\phi_2$ and $\dim r\phi_2/V + V^\perp$).

4. Dense embeddings of finite codimension

Here we consider embeddings which are not assumed to be of finite type. Our aim is to set up a congruence theorem for $\perp$ -dense embeddings E of a fixed finite codimension $m = \dim E/V < \infty$. By virtue of the normal form (3) of (V, ϕ) there is no loss of generality in assuming that V is stable with $\|V\| = \bigcap_{\mu \in I} P_\mu$ for some $I \subset X(k)$. As explained in Section 2 the spaces $C_\mu(V)$ and maps $(\phi_V)_\mu : C_\mu(V) \times C_\mu(V) \longrightarrow \mathbb{R}$ ($\mu \in I$) lack sufficient content of information, in contrast to the $\mathbb{R}$ -matrices ${}^\mu B$ together with the k -matrix $\phi(x_i, x_k)_{1 \leq i, k \leq m}$ (both are defined with respect to a basis $x_1, \dots, x_m$ of a supplement X of V in E). The drawback of these quantities lies in their unwieldy transformation law. Some normalization can be obtained by requiring $x_1, \dots, x_m$ to be an orthonormal system. To prove the existence of such $x_1, \dots, x_m$ use the following remark: If $x \in E \setminus V$ there is $\lambda \in \mathbb{Q}$, $0 < \lambda < 1$ such that $\phi(\lambda x, \lambda x) < 1$ for all $\mu \in I$; hence there exists $v \in V \cap A^\perp$ such that

$\Phi(\lambda x + v, \lambda x + v) = 1$ ($A \subset E$ any finite dimensional subspace). But how do the ${}^H B$ transform if we switch from one orthonormal system to another? If $x'_1 \equiv \sum_K S_{1K} x_K \pmod{V}$ then, for all sufficiently large n , we have (cf. Section 2 for the definitions)

$$(5) \quad A'_n - 1 = S(A_n - 1)S^*.$$

Solving for $B'_n = A_n'^{-1}$ we find

$$\begin{aligned} B'_n &= [1 + S^{*-1} B_n S^{-1} - S^{*-1} B_n S^*]^{-1} \cdot S^{*-1} B_n S^{-1} \\ &= \Delta(S, B_n)^{-1} \cdot S^{*-1} B_n S^{-1} \end{aligned}$$

where the factor $\Delta(S, B_n)^{-1}$ is also equal to $1 + B'_n S S^* - B'_n$ since $1 + B'_n S S^* - B'_n = B'_n S A_n S^*$ (multiply (5) on the left by B'_n). Now apply $\mu \in I$ to these equations and let n tend to infinity. Viewing $k \rightarrow \mathbb{R}^I$ as an inclusion and $B = ({}^H B)_{\mu \in I}$ as a matrix over $\mathbb{R}^I$ we find that $\Delta(S, B)$ is invertible and

$$(6) \quad B' = \Delta(S, B)^{-1} \cdot S^{*-1} B S^{-1}.$$

We have proved one half of

Theorem 3. Let $V \subset (E, \Phi)$ be $\perp$ -dense with $E = V \oplus kx_1 \oplus \dots \oplus kx_m$, V stable, $\|V\| = \bigcap_{\mu \in I} P_\mu$, $\Phi(x_1, x_K) = \delta_{1K}$ and consider a second situation $V' \subset (E', \Phi')$, $x'_1, \dots, x'_m$ of the same kind, with $V' \equiv V$. In order that the two embeddings be congruent it is necessary and sufficient that there exist an invertible k -matrix S such that $\Delta(S, B) := 1 + S^{*-1} B S^{-1} - S^{*-1} B S^*$ is an invertible $\mathbb{R}^I$ -matrix and (6) holds.

In order to prove the other half it suffices, by Lemma 1, to exhibit orthonormal systems $\bar{x}_1, \dots, \bar{x}_m$ in E ; $\bar{x}'_1, \dots, \bar{x}'_m$ in E' for which $\bar{B} = \bar{B}'$. Set $y_1 := \sum_K S_{1K} x_K$ where S is the matrix in the theorem. Then there are $v_1 \in V$ such that the $y_1 + v_1$ are orthogonal to each

other; furthermore there are $\lambda \in \mathbb{Q}$, $0 < \lambda < 1$ and $w_1 \in V$ such that the $\bar{x}_1 := \lambda(y_1 + v_1) + w_1$ form an orthonormal system, further $w'_1 \in V'$ such that the $\bar{x}'_1 := \lambda x'_1 + w'_1$ form an orthonormal system ($i = 1, \dots, m$). Computing $\bar{B}$ and $\bar{B}'$ from equations analogous to (6) and using (6) itself we find $\bar{B} = \Delta(\lambda S, B)^{-1} (\lambda S)^{-1} B (\lambda S)^{-1} = \Delta(\lambda S, B)^{-1} (\lambda^{-2} - 1) \Delta(S, B) B' = \Delta(\lambda S, B)^{-1} \Delta(S, B) \Delta(\lambda^{-1} B) \bar{B}'$. Hence it is enough to show that $\Delta(\lambda S, B) = \Delta(S, B) \cdot \Delta(\lambda^{-1} B')$. Using once more that B, B', S are related by (6) the right hand side becomes $\Delta(S, B) (1 + (\lambda^{-2} - 1) \Delta(S, B)^{-1} S^{*-1} B S^{-1}) = \Delta(S, B) + (\lambda^{-2} - 1) S^{*-1} B S^{-1} = 1 + \lambda^{-2} S^{*-1} B S^{-1} - S^{*-1} B S^{-1} = \Delta(S, B)$. Q.E.D.

Theorem 3 may be supplemented by the following remark which we mention without proof (the construction in II.2.4 needs several refinements): If $I \subset k(X)$ and for each $\mu \in I$ we are given a positive semidefinite $m \times m$ -matrix ${}^\mu B$ over $\mathbb{R}$ then there is an embedding $V \subset (E, \phi)$ of the kind considered in the theorem which takes on the given $\mathbb{R}^I$ -matrix $B = ({}^\mu B)_{\mu \in I}$ with respect to a suitable orthonormal system $x_1, \dots, x_m$. (Again this fact rests upon the density of k in $\mathbb{R}^I$, i.e. on the Artin-Whaples approximation theorem.)

The reader is invited to check that in case $k = \mathbb{R}$ Theorem 3 is in accordance with I.4 Theorem 1: By a transformation (6) the matrix B may be brought to diagonal form with diagonal entries $b_i \in \{0, \frac{1}{2}, 1, 2\}$. Then the number of b_i 's which equal $0, \frac{1}{2}, 1, 2$ coincides with $\dim E/C(V)$, $n^+(C(V), \phi_V)$, $\dim r \phi_V/V$, $n^-(C(V), \phi_V)$ respectively.

5. Extending isometries; orthogonal separation

When generalizing the results of Chapter VI one has to face the same difficulties as in the previous sections. For the sake of simplicity we shall here content ourselves with the case of spaces (E, ϕ) which admit orthonormal bases (otherwise the statements would have to

be restricted to subspaces meeting certain finiteness conditions, similarly to Section 3). Thus for each $V \subset E$ the symmetric k -bilinear map $\psi = \phi_{V^\perp} : E/V^\perp \times E/V^\perp \longrightarrow \mathbb{R}^{X(k)}$ is defined, and a straightforward adaptation of the proofs in Chapter VI yields the following theorems.

Theorem 4 . Let (E, ϕ) , (E', ϕ') have orthonormal bases and let $\varphi : V \xrightarrow{\sim} V'$ be an isometry between $\perp$ -closed subspaces $V \subset E$, $V' \subset E'$. Assume that $\dim V^\perp = \dim V'^\perp = \aleph_0$. In order that φ extend to an isometry $E \xrightarrow{\sim} E'$ it is necessary and sufficient that φ be weakly homeomorphic and $\varphi^\perp : E'/V'^\perp \longrightarrow E/V^\perp$ transform ψ' into ψ .

Theorem 5 . Let $A \perp B$ be an orthogonal pair of subspaces in a space (E, ϕ) with orthonormal basis. In order that A, B be orthogonally separated in E it is necessary and sufficient that $(A+B)^{\perp\perp} = A^{\perp\perp} + B^{\perp\perp}$, $A^\perp + B^\perp = E$, and, for each $\mu \in X(k)$, $x \in A^\perp$, $y \in B^\perp$,

$$\mu(\phi(x, y)) = \sum_{i \in I} \mu(\phi(x, v_i) \phi(v_i, v_i)^{-1} \phi(v_i, y))$$

where $(v_i)_{i \in I}$ is an arbitrary orthogonal basis of $A^\perp \cap B^\perp$.

APPENDIX

WILDNESS OF A CERTAIN CLASSIFICATION PROBLEM

In our investigations on the congruence problem we have - on quite different occasions - chanced two or three times upon the following situation. There is given a fixed vector space Q of infinite dimension over its base field k (which, for the sake of simplicity, is assumed here commutative), and one would like to classify symmetric k -bilinear maps

$$\Psi : F \times F \longrightarrow Q$$

on k -spaces F of finite or countably infinite dimension, with respect to isomorphism: Two such objects are called isomorphic, $(F, \Psi) \approx (F', \Psi')$, if there exists an isomorphism $T: F \xrightarrow{\sim} F'$ such that $\Psi'(Tx, Ty) = \Psi(x, y)$ for all $x, y \in F$. In IV.2.3 we had $k = \mathbb{R}$ and $Q = \mathbb{R}^{\mathbb{N}} / \mathbb{R}^{(\mathbb{N})}$ whereas in VII.3 we had $k \subset \mathbb{R}$ (a proper subfield in order to exclude trivialities) and $Q = \mathbb{R}^I$ (where $I \subset X(k)$) or $Q = \mathbb{R}^{I_1/k} \times \mathbb{R}^{I_2/k} \times \mathbb{R}^{I_1 \cap I_2}$. In this appendix we shall try to make plausible¹⁾ that the above classification problem is "wild" in the sense that it is at least as complicated as any classification problem out of a certain wide range of problems. We then also say that the objects (F, Ψ) are "unclassifiable". Before beginning our explanations we remark that they will by necessity be accompanied by the vagueness associated with the term "classification".

The term "wild" is commonly used in the representation theory of algebras, where one would like to classify all modules of finite k -dimension over a given k -algebra A . Although nobody has dared to give a definition up to now there are algebras which are definitely, in

¹⁾ to the uninitiate reader; the reader who is familiar with these ideas will not find anything new here.

everybody's opinion, of wild representation type. As an example we may adduce the free algebra $k\langle X, Y \rangle$ generated by two elements X, Y . It has the striking property that for each finitely generated k -algebra A there exists a full faithful functor

$$F_A : \text{mod } A \longrightarrow \text{mod } k\langle X, Y \rangle$$

($\text{mod } A$ is the category of k -finite dimensional left A -modules) so that a complete classification of all $k\langle X, Y \rangle$ -modules would also imply a classification of all A -modules, for each k -algebra A of finite type. It is easy to define such a functor F_A : If A is generated by $a_1, \dots, a_n$, put $F_A(M) = M^{n+2}$ for each $M \in \text{mod } A$, where X, Y operate on M^{n+2} by means of the matrices

$$\begin{pmatrix} 0 & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & 1 \\ 0 & & & & & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & & & 0 \\ 1 & \cdot & & & \\ a_1 & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ 0 & & a_n & 1 & 0 \end{pmatrix}$$

As a further example, important to our purpose, consider the algebra belonging to the quiver

$$K = \begin{array}{ccc} & \xrightarrow{\alpha_1} & \\ \bullet & \vdots & \bullet \\ 1 & \xrightarrow{\alpha_n} & 2 \end{array}$$

(two points with n arrows between them) where n is supposed to be ≥ 3 . We have no need to describe this algebra explicitly because its modules correspond naturally to the representations of the quiver. (The facts on quivers and their representations needed here can be found in [8].) A representation G of K is written as $G = (G_1, G_2; \varphi_1, \dots, \varphi_n)$ where G_1, G_2 are k -spaces and the φ_i are linear maps $G_1 \rightarrow G_2$. If $\text{mod } K$ is the category of finite dimensional representations of K

we have a full faithful functor

$$\begin{array}{ccc} \text{mod } k \langle X, Y \rangle & \longrightarrow & \text{mod } K \\ M & \longmapsto & (M, M; X^{\cdot}, Y^{\cdot}, \text{id}_M, \text{id}_M, \dots) \end{array}$$

(it is full because of $n \geq 3$) which shows that, again, the representations of K are unclassifiable.

The reader may wonder what our quadratic objects (F, Ψ) should have to do with the representations of the quiver K . The connection is provided by the theory of quadratic forms in additive categories by Quebbemann - Scharlau - Scharlau - Schulte [19], [20]. In order to be able to apply their theory we have to restrict ourselves to the (F, Ψ) with F of finite dimension (these, too, turn out to be unclassifiable). Then the image of $F \times F$ is already contained in a finite dimensional subspace $Q_1 \subset Q$. Fixing one such subspace Q_1 of dimension $n \geq 3$ let us further restrict our attention to the (F, Ψ) with $\Psi(F \times F) \subset Q_1$. Finally, by fixing an isomorphism $Q_1 \cong k^n$ we may interpret (F, Ψ) as $(F, \Psi_1, \dots, \Psi_n)$, a vector space F endowed with n symmetric forms. Now let $\mathfrak{B}$ be the category of these objects $(F, \Psi_1, \dots, \Psi_n)$ with the isomorphisms defined at the beginning as morphisms. By [19], 4.3 the category $\mathfrak{B}$ is equivalent to the category $\mathfrak{K}(\text{mod } K)$ of nonsingular symmetric forms in $\text{mod } K$. Let us explain this: The category $\text{mod } K$ is equipped with a duality functor $*$ defined by $(G_1, G_2; \varphi_1, \dots, \varphi_n)^* := (G_2^*, G_1^*; \varphi_1^*, \dots, \varphi_n^*)$ where for a k -space X of finite dimension X^* is its dual space. This functor $*$ enjoys properties analogous to those of the duality functor for k -spaces; in particular one may identify each $G \in \text{mod } K$ with its bidual G^{**} by means of a canonical isomorphism $G \xrightarrow{\sim} G^{**}$. This shall always be done in the sequel. Then, a nonsingular symmetric form on $G \in \text{mod } K$ is an isomorphism $b: G \rightarrow G^*$ with $b = b^*$. An isometry $(G, b) \rightarrow (G', b')$ is an isomorphism $g: G \rightarrow G'$

such that $b = g^* b' g$. The nonsingular symmetric forms (G, b) are the objects of the category $\mathcal{K}(\text{mod } K)$; its morphisms are the isometries. The equivalence of categories

$$\mathfrak{B} \xrightarrow{\sim} \mathcal{K}(\text{mod } K)$$

mentioned before is given by

$$(F, \psi_1, \dots, \psi_n) \longmapsto (G, b)$$

where $G := (F, F^*; \psi_1, \dots, \psi_n)$, $\psi_i : F \longrightarrow F^*$, $\psi_i(x)(y) = \psi_i(x, y)$, $b := (\text{id}_F, \text{id}_{F^*})$.

Thus, our question on the wildness of $\mathfrak{B}$ translates to the corresponding question on $\mathcal{K}(\text{mod } K)$. It is more or less evident that a classification of the objects of $\mathcal{K}(\text{mod } K)$ would only be possible on grounds of a classification of the objects of the underlying category $\text{mod } K$ (which is, however, of wild type, as we have seen above). In order to render this evidence more concrete we consider the "neutral" functor

$$\begin{aligned} v : \text{mod } K &\longrightarrow \mathcal{K}(\text{mod } K) \\ M &\longmapsto v(M) := (M \oplus M^*, \begin{pmatrix} 0 & \text{id}_{M^*} \\ \text{id}_M & 0 \end{pmatrix}) \\ (g: M \rightarrow M') &\longmapsto v(g) := \begin{pmatrix} g & 0 \\ 0 & g^{*-1} \end{pmatrix} \end{aligned}$$

(it is not defined on all morphisms in $\text{mod } K$, merely on isomorphisms). If M, M' are indecomposable in $\text{mod } K$ and $v(M) \cong v(M')$ then either $M \cong M'$ or $M^* \cong M'$. This shows that a classification of the neutral forms, i.e., those in the image of v , let alone the others, would presuppose the knowledge of the indecomposable objects of $\text{mod } K$ (and hence of all objects) together with the action of the duality functor.

REFERENCES

- [1] W. Allenspach, Erweiterung von Isometrien in alternierenden Räumen. Ph. D. Thesis, University of Zurich, 1973.
- [2] W. Bäni, Subspaces of positive definite inner product spaces of countable dimension. Pac. J. Math. 82, 1-14 (1979).
- [3] —, Inner product spaces of infinite dimension; on the lattice method. Arch. Math. 33, 338-347 (1979).
- [4] W. Bäni, H. Gross, On SAP fields. Math. Z. 162, 69-74 (1978).
- [5] G. Birkhoff, Lattice Theory. 3rd edition 1973, AMS.
- [6] J. Bognár, Indefinite Inner Product Spaces. Ergebnisse Band 78, Springer, Berlin - Heidelberg - New York 1974.
- [7] L. Brand, Erweiterung von algebraischen Isometrien in sesquilinearen Räumen. Ph. D. Thesis, University of Zurich, 1974.
- [8] P. Gabriel, Représentations indécomposables. Sémin. Bourbaki Exp. 444, in Springer Lecture Notes Vol. 431, 1973/74.
- [9] H. Gross, On Witt's Theorem in the denumerably infinite case. Math. Ann. 170, 145-165 (1967).
- [10] —, Eine Bemerkung zu dichten Unterräumen reeller quadratischer Räume. Comment. Math. Helv. 45, 472-493 (1970).
- [11] —, Isomorphisms between lattices of linear subspaces which are induced by isometries. J. Alg. 49, 537-546 (1977).
- [12] —, Quadratic Forms in Infinite Dimensional Vector Spaces. Progr. in Math. Vol. 1, Birkhäuser, Boston - Basel - Stuttgart 1979.
- [13] B. Jónsson, Distributive sublattices of a modular lattice. Proc. Amer. Math. Soc. 6, 682-688 (1955).
- [14] I. Kaplansky, Forms in infinite dimensional spaces. An. Acad. Brasil. Cienc. 22, 1-17 (1950).

- [15] G. Köthe, Topological Vector Spaces I. Springer, Berlin - Heidelberg - New York 1969.
- [16] G. Maxwell, Classification of countably infinite hermitean forms over skew fields. Amer. J. Math. 96, 145-155 (1974).
- [17] R. Moresi, Untersuchungen in abzählbar-dimensionalen nicht spurwertigen ϵ -hermiteschen Räumen. Ph. D. Thesis, University of Zurich, 1980.
- [18] A. Prestel, Remarks on the Pythagoras and Hasse number of real fields. J. reine angew. Math. 303/304, 284-294 (1978).
- [19] H.G. Quebbemann, R. Scharlau, W. Scharlau, M. Schulte, Quadratische Formen in additiven Kategorien. Bull. Soc. Math. France 48, 93-101, (1976).
- [20] H.G. Quebbemann, W. Scharlau, M. Schulte, Quadratic and hermitean forms in additive and abelian categories. J. Alg. 59, 264-289 (1979).
- [21] U. Schneider, Beiträge zur Theorie der sesquilinearen Räume unendlicher Dimension. Ph. D. Thesis, University of Zurich, 1975.
- [22] E. Witt, Theorie der quadratischen Formen in beliebigen Körpern. J. reine angew. Math. 176, 31-44 (1937).

Symbols

(E, t)	5	$V_1(E)$	29
V, R	5	$\mathcal{E}_1$	30
V_-, V_+	5	$V_{red}(E)$	34
$n^-(V), n^+(V)$	5	V, V_{red}	38
E	5	Z_n, Z	39
$C(V), \mathcal{E}_V, r_V^{\mathcal{E}}$	6	$v(E)$	45
$\Phi V \Phi$	7	$\mathcal{E}_n, \mathcal{E}_n, \dots, \mathcal{E}_n$	47
G	10	$\tau(A, t, B)$	59, 88
τ	10	τ^t	89
$L(E)$	11	E	100
$M(X, Y), D(X, Y)$	11	$-k^2$	100
F	12	P_L	100
A_n, B_n, B	17, 18, 102	$X(k)$	100
$V(E)$	27	$C(V)$	101
E_i, V_i, F_i	27	B	102
E_{α}, V_{α}	27	$\cdot$	103
$\mathcal{E}_1, \mathcal{F}_1$	27	$\mathcal{E}$	106
R_i, S_i	28		

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